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Matrix Models an Knot Theory

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Summary by Dominique Gouyou-Beauchamps

Abstract

We shall explain how knot, link and tangle enumeration problems can be expressed as matrix integrals which will allow us to use quantum field-theoretic methods. We shall discuss the asymptotic behaviors for a great number of intersections. We shall detail algorithms used to test our conjectures.

1. Classification and Enumeration of Knots, Links, Tangles

A knot is defined as a closed, non-self-intersecting curve that is embedded in three dimensions and cannot be untangled to produce a simple loop (i.e., the unknot). A knot can be represented by its plane projection (i.e., its diagram). A knot can be generalized to a link, which is simply a knotted collection of one or more closed strands. A tangle is defined as a region in a knot or link projection plane surrounded by a circle such that the knot or link crosses the circle exactly four times. An alternating knot (resp. link) is a knot (resp. link) which possesses a knot diagram (resp. link diagram) in which crossings alternate between under- and overpasses (see Figure 1).

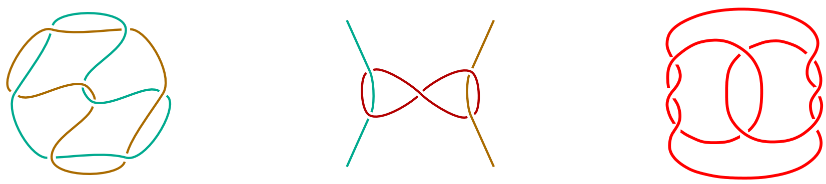


Figure 1. An alternating link, a tangle and an non-alternating knot

Figure 2. A 6_1 knot of the Tait's classification

P. G. Tait [16, 17, 18, 19, 20, 21] undertook a study of knots in response to Kelvin's conjecture that the atoms were composed of knotted vortex tubes of ether (Thompson [24]). He categorized knots in terms of the number of crossings in a plane projection (see Figure 2). He also made some conjectures which remained unproven until the discovery of Jones polynomials:

1. Reduced alternating diagrams have minimal link crossing number,
2. Any two reduced alternating diagrams of a given knot have equal writhe,
3. The flyping conjecture, which states that the number of crossings is the same for any reduced diagram of an alternating knot (see [25] for definition of the flyping equivalence).

Conjectures (1) and (2) were proved by Kauffman [4], Murasugi [9], and Thistlethwaite [22, 23] using properties of the Jones polynomial or Kauffman polynomial **F** (see Hoste et al. [1]). Conjecture (3) was proved true by Menasco and Thistlethwaite [7, 8] using properties of the Jones polynomial. Schubert [12] showed that every knot can be uniquely decomposed (up to the order in which the decomposition is performed) as a knot sum of a class of knots known as prime knots, which cannot themselves be further decomposed. Knots that are the sums of prime knots are known as composite knots.

There is no known formula for giving the number of distinct prime knots as a function of the number of crossings. The numbers of distinct prime knots having $n = 1, 2, \ldots$ crossings are 0, 0, 1, 2, 3, 7, 21, 49, 165, 552, 2176, 9988, ... (Sloane's M0851) [13].

In the 1932, Reidemeister [10] first rigorously proved that knots exist which are distinct from the unknot. He did this by showing that all knot deformations can be reduced to a sequence of three types of "moves," called Reidemeister moves (see Figure 3).

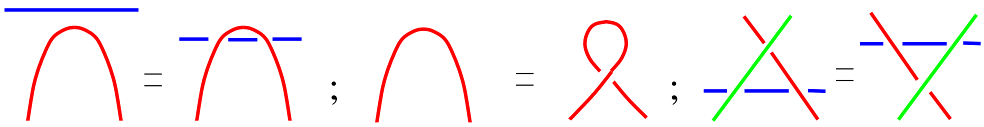


FIGURE 3. The three Reidemeister moves (poke, twist and slide)

2. Feynman Diagrams. $O(n)$ Matrix Model and Renormalization

We want to enumerate prime alternating tangles with a given number of components and crossing. Let $a_{k;p}$ be the number of prime alternating tangles with $k+2$ components and p crossings. Let $\Gamma(n,g)$ be the corresponding generating series:

$$\Gamma(n,g) = \sum_{k=0,p=1}^{\infty} a_{k;p}\, n^k g^p$$

In [29], we have shown that the integral

$$Z = \int dM e^{N[-\frac{1}{2}trM^2 + \frac{g}{4}trM^4]} \quad \text{where} \quad dM = \prod_i dM_{ii} \prod_{i<j} d\Re e M_{ij}\, d\Im m M_{ij} \tag{1}$$

over $N \times N$ hermitean matrices is well suited for the counting of alternating links and tangles: for an appropriate choice of $\alpha(g)$, $2\frac{\partial}{\partial g} \lim_{N\to\infty} \frac{1}{N^2} \log Z_n(g, \alpha(g))$ is the generating function of the number of alternating tangle diagrams with n 4-valent crossings. In the context of knot theory, it seems natural to consider this integral over complex (non hermitean) matrices, in order to distinguish between under-crossing and over-crossing. However, this integral is closely related to the simpler integral (1) in the large N limit.

It has been known to physicists since the pioneering work of 't Hooft [15] that the large N limit of the previous integral (1) may be organized in a topological way. While the leading term corresponds to planar diagrams, the subdominant terms of order N^{-2h} of $\frac{1}{N^2}\log Z(g)$ are describe by graphs drawn on a Riemann surface of genus h.

The series expansion of Z in powers of g may be represented diagrammatically by Feynman diagrams, made of undirected edges or "propagators" with double lines expressing the conservation of indices, $\langle M_{ij}M_{kl}\rangle_0 = \frac{1}{N}\delta_{il}\delta_{jk}$ and the 4-valent vertices $gN\delta_{qi}\delta_{jk}\delta_{lm}\delta_{np}$ (see Figure 4). More precisely

$$\lim_{n\to\infty}\frac{1}{N^2}\log Z = \sum \text{ weight } g^n$$

where the sum is over all planar diagrams with n vertices and with a weight equal one over the order of the automorphism group of the diagram.

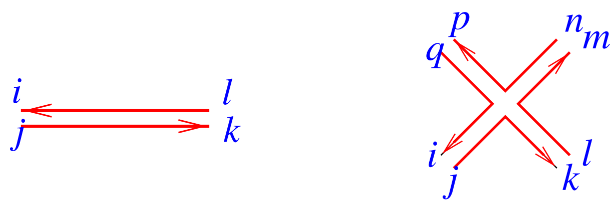


FIGURE 4. Propagator and 4-valent vertex

In order to connect integral (1) with knot theory, we take any planar diagram (i.e., 4-regular planar map) and to the following: starting from an arbitrary crossing, we decide it is a crossing of two strings (again there is an arbitrary choice of which is under/over-crossing). Once the first choice is made, we simply follow the string and form alternating sequences of under- and over-crossings. The remarkable fact is that this can be done consistentlty (see Figure 5). If we identify two alternating diagrams obtained from one another by inverting undercrossings and overcrossings, then there is a one-to-one correspondence between planar diagrams and alternating link diagrams.

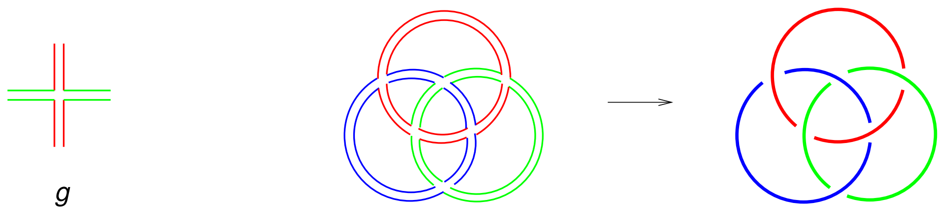


FIGURE 5. A planar diagram and one of the two corresponding alternating link diagrams

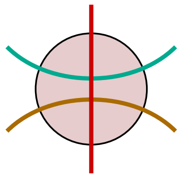
In order to distinguish strands of links, we introduce a more general model, which we shall call the intersecting loops $O(n)$ model. If n is a positive integer, then consider the following multi-matrix integral:

$$Z^{(N)}(n,g) = \int \prod_{a=1}^{n} d\,M_a\, e^{N\,tr\left(-\frac{1}{2}M_a^2+\sum_{b=1}^{n}\frac{g}{4}(M_aM_b)^2\right)} \tag{2}$$

and the corresponding free energy

$$F(n,g) = \lim_{N\to\infty}\frac{\log Z^{(N)}(n,g)}{N^2} = \sum_{k,p=1}^{\infty} f_{k;p}n^k g^p.$$

The correlation functions count tangle diagrams:

$$\lim_{N\to\infty}\left\langle \frac{1}{N} tr(M_1M_2M_3M_2M_1M_3)\right\rangle_c =$$  .

For tangles, a convenient way to keep track of the number of connected components and of the connections of the external legs is to use colors. The colors allow us to distinguish the various external legs and add an extra power series variable in the theory (the number of colors n) to count separately objects with different numbers of connected components.

This model has two problems:

1. the diagrams generated by applying Feynman rules are not necessarily reduced or prime,
2. several reduced diagrams may correspond to the same knot due to the flyping equivalence.

We are therefore led to renormalize the quadratic and quartic interaction of (2). A key observation is that, while there is only one such quadratic $O(n)$-invariant term, there are two quartic $O(n)$-invariant terms, which leads to a generalized model with 3 coupling constants (i.e., t, g_1 and g_2) in the action (bare coupling constants):

(3) $$Z^{(N)}(n,t,g_1,g_2) = \int \prod_{a=1}^{n} d\,M_a e^{Ntr\left[-\frac{t}{2}M_a^2+\sum_{b=1}^{n}\left(\frac{g_1}{4}M_aM_bM_aM_b+\frac{g_2}{2}M_aM_aM_bM_b\right)\right]}$$

where t, g_1 and g_2 are functions of the renormalized coupling constant g, chosen such that the correlation functions are the appropriate generating series in g of the numbers of alternating links (see [28] and Figure 6).

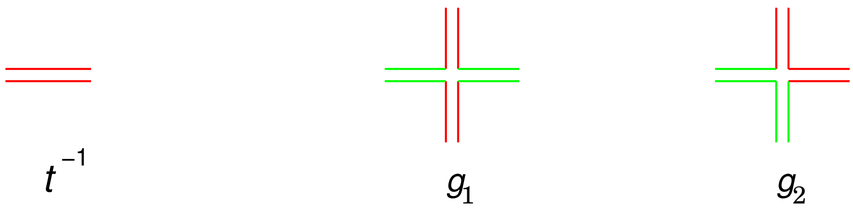


FIGURE 6. The three coupling constants

There are currently two values of n for which the corresponding matrix model has been exactly solved: $n = 1$ and $n = 2$.

The case $n = 1$ is particulary important since it corresponds to counting all alternating tangles regardless of the number of connected components. We have the usual matrix model

$$Z^{(N)}(t,g) = \int dM\, e^{N\,tr\left(-\frac{t}{2}M^2+\frac{g}{4}M^4\right)} \quad \text{with } g = g_1 + 2g_2 \ .$$

"Renormalization" equations recombine into a fifth degree equation:

$$32 - 64\,A + 32\,A^2 - 4\frac{1+2g-g^2}{1-g}A^3 + 6\,g\,A^4 - g\,A^5 = 0.$$

Correlation functions are given in terms of its solution. In particular, if $\left\langle \frac{1}{N}trM^{2\ell}\right\rangle_c = \sum_{p=0}^{\infty} a_p g^p$ is the generating function of prime alternating tangles with 2ℓ legs where p counts the crossings, then

$$a_p \overset{p\to\infty}{\sim} \text{cst}\, g_c^{-p}\, p^{-5/2} \quad \text{with } g_c = \frac{\sqrt{21001}-101}{270} \ (g_c^{-1} \approx 6.147930)\ .$$

These works generalize known results: the case $\ell = 2$ (Sundberg and Thistlethwaite [14]), and Zinn-Justin and Zuber [29]. The number f_p of prime alternating links grows like $f_p \sim \mathrm{cst}\, g_c^{-p} p^{-7/2}$ (Kunz-Jacques and Schaeffer [6]).

In case $n = 2$, integral (2) is equivalent to integral (4) that is recently studied in detail and computed in the framework of the random lattice model (Zinn-Justin [26] and Kostov [5]). In [30], we thus carry out the explicit counting of alternating 2-color tangles: their generating function is the solution of coupled equations involving elliptic functions.

$$Z^{(N)}(t, g_1, g_2) = \int dM_1\, dM_2\, e^{N\,tr\left[-\frac{t}{2}(M_1^2+M_2^2)+\frac{g_1+2g_2}{4}(M_1^4+M_2^4)+\frac{g_1}{2}(M_1M_2)^2+g_2M_1^2M_2^2\right]} \tag{4}$$

When we introduce a complex matrix $X = \frac{1}{\sqrt{2}}(M_1 + iM_2)$, we obtain:

$$Z^{(N)}(t, b, c) = \int dX\, dX^\dagger\, e^{N\,tr\left(-tXX^\dagger+bX^2X^{\dagger 2}+\frac{1}{2}c(XX^\dagger)^2\right)} \quad \text{with } b = g_1 + g_2 \text{ and } c = 2g_2\ .$$

The number γ_p of prime alternating 2-color tangles with p crossings grows like

$$\gamma_p \overset{p\to\infty}{\sim} \mathrm{cst}\, g_c^{-p} p^{-2} (\log p)^{-1} \quad \text{with } g_c^{-1} \approx 6.28329764\ .$$

In [27], we establish that the number f_p of reduced alternating link diagrams with two colors and p crossings has the following asymptotics:

$$f_p \overset{p\to\infty}{\sim} \mathrm{cst}\, g_c^{-p}\, p^{-3} \log p \quad \text{with } g_c = \frac{\pi(\pi-4)^2}{16}\ (g_c^{-1} \approx 6.91167)\ .$$

The number $g_c^{-1} = 6.91167\ldots$ is slightly larger than the value 6.75 obtained for only one color.

3. Algorithm: Transfer Matrix

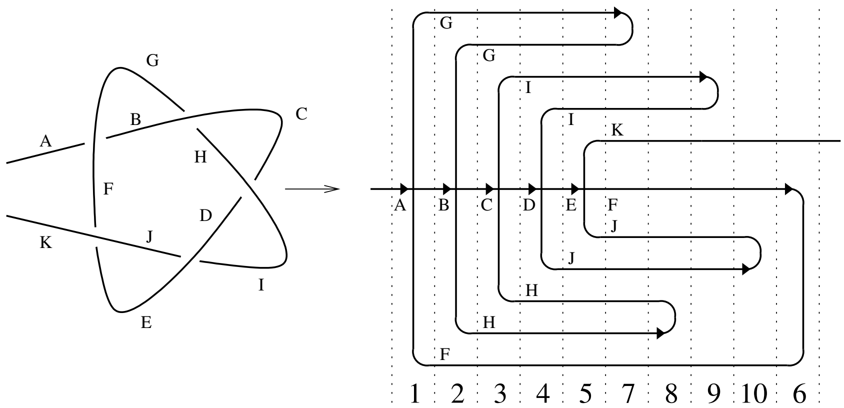


Figure 7. The steps are ordered by their number below the diagram. At each step, the active line is distinguished by an arrow.

In [2, 3], we propose a new method to enumerate alternating knots using a transfer matrix approach. The basic ingredient is the ability to cut the object one is studying into slices, which represent the state of the system a fixed (discrete) time. The naive idea would be to draw the knot diagrams on the plane in such a way that time would correspond to one particular coordinate of the plane, that is to read the knot diagrams "from left to right." Here, this idea does not work directely, and one is led to a slightly more sophisticated notion of slices, which we shall explain using the example of Figure 7.

A basis state will be described by a series of left and right arches and the position of the active line. As an illustration, we show all the intermediate states of the example of Figure 7 on Figure 8.

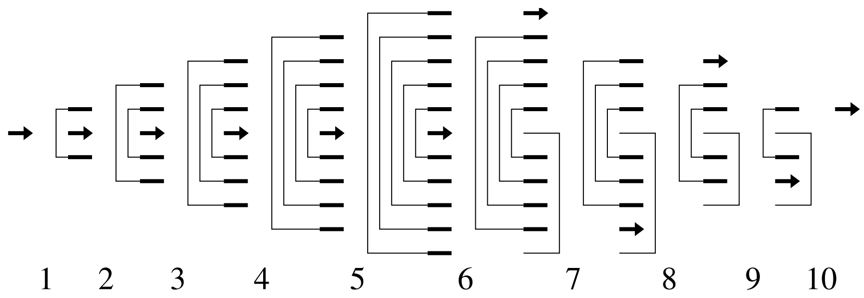


FIGURE 8. A sequence of intermediate states. The active line is denoted by an arrow.

The table below gives numbers $a_{k,p}$ of prime alternating tangles with 2 external legs, k circles (i.e., $k+1$ connected components) and p crossings up to $p = 15$ (even though they can be easily obtained for p up to 18 or 19, as in [3], on a work station, and probably further using larger computers). Tangles of types 1 (i.e., Γ_1) and 2 (i.e., Γ_2) are distiguished by the two ways of connecting their external legs. The reader is reminded that the total number of tangles is given by $\Gamma_1 + 2\Gamma_2$.

	Γ_1							Γ_2						
p \ k	0	1	2	3	4	5	6	0	1	2	3	4	5	6
1	1							0						
2	0							1						
3	2							1						
4	2							3	1					
5	6	3						9	1					
6	30	2						21	11	1				
7	62	40	2					101	32	1				
8	382	106	2					346	153	24	1			
9	1338	548	83	2				1576	747	68	1			
10	6216	2968	194	2				7040	3162	562	43	1		
11	29656	11966	2160	124	2			31556	17188	2671	121	1		
12	131316	71422	9554	316	2			153916	80490	15295	1484	69	1	
13	669138	328376	58985	5189	184	2		724758	425381	87865	6991	194	1	
14	3156172	1796974	347038	22454	478	2		3610768	2176099	471620	52231	3280	103	1
15	16032652	9298054	1864884	193658	10428	260	2	17853814	11376072	2768255	308697	15431	290	1

4. **Algorithm: Random Sampling**

In [11], in order to generate a random map according to the uniform distribution on rooted 4-regular planar maps with p vertices one generate a blossom tree according to the uniform distribution on blossom tree and apply closure that is a $(p+2)$-to-2 correspondence between blossom trees and rooted 4-regulat maps (see Figure 9). This algorithm allows to generate in linear time (up to $p = 10^7$ vertices) rooted 4-regular planar maps with p vertices and two legs, with uniform probability. One can compute various quantities related to the map thus generated and then average over a sample, as always in Montecarlo simulations.

The main idea of the physical interpretation of the number $a_p(1)$ of rooted 4-regular maps is to consider planar maps as discretized random surfaces (with the topology of the sphere). As the

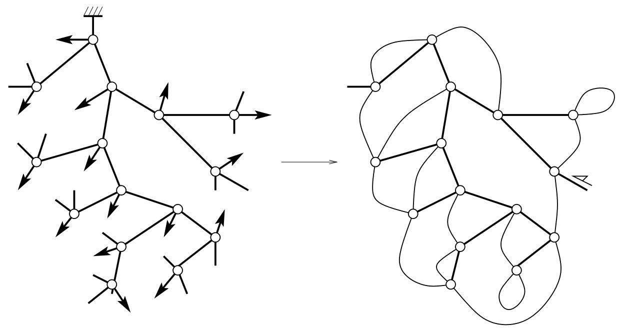


FIGURE 9. Schaeffer's bijection between blossom trees and planar maps.

number of vertices of the map grows large, the details of the discretization can be assimilated to the fluctuations of the metric on the sphere. Now, physics tell us that the metric is the dynamical field of general relativity, i.e., gravity, and that this type of fluctuations in the metric are characteristic of a quantum theory. In our case it means that, as p becomes large, the discrete nature of the maps can be ignored and there exists a scaling limit, the properties of which are described by two-dimensional euclidian gravity. In particular any parameter of random planar maps that makes sense in the scaling should converge to its continuum analog. A fundamental parameter of this kind turns out to be precisely the number of (unrooted) planar maps: it is expected to scale to the partition function $Z_g(A)$ of two-dimensional quantum gravity with spherical topology at fixed area A, through a relation of the form $\frac{1}{p}a_p(1) \overset{p\to\infty}{\sim} Z_g(A)$, with A proportional to p.

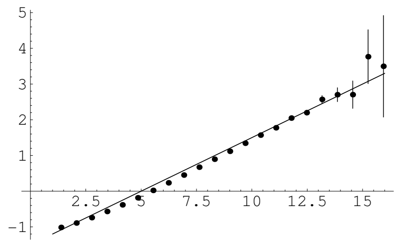


FIGURE 10

Here the factor $1/p$ is due to the fact that the partition function does not takes the rooting into account. We conjecture that for $|n| < 2$, the matrix model is in the universality class of a 2D field theory with spontaneously broken $O(n)$ symmetry, coupled to gravity. The large size limit is described by a a conformal field theory (CFT) coupled to gravity with $c = n - 1$:

$$a_p(n) \sim \mathrm{cst}(n)\, g_c(n)^{-p}\, p^{\gamma(n)-2}, \qquad f_p(n) \sim \mathrm{cst}(n)\, g_c(n)^{-p}\, p^{\gamma(n)-3},$$
$$\gamma = \frac{c - 1 - \sqrt{(1-c)(25-c)}}{12}.$$

In particular, knots correspond to the limit $n \to 0$:

$$f_p(0) \sim \mathrm{cst}\, g_c^{-p} p^{-\frac{19+\sqrt{13}}{6}}$$

With Schaeffer's algorithm, we have tested quantity: $\gamma' \equiv \frac{d\gamma}{dn}_{|n=1} = 3/10$ according to the conjecture. We obtain a very good agreement (see Figure 10).

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