

On the Asymptotic Analysis of a Class of Linear Recurrences

Thomas Prellberg

Technische Universität Clausthal (Germany)

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Summary by Marni Mishna

Abstract

This work offers an interesting application of analytic iteration theory and classical complex analysis to determine some new (and old) results in asymptotic enumeration. The method treats functional equations of a particular form, which have a natural interpretation in terms of combinatorial generating functions. Partition lattice chains and Takeuchi numbers are among the applications of this method presented here.

1. Problems Suited to this Analysis

Many combinatorial classes can be described in a recursive way, built from basic atomic units using a handful of combinatorial operations, as described in [2]. One of the principal fruits of this point of view is a set of functional equations for the exponential and ordinary generating functions of the family. The work presented here considers families satisfying a particular type of combinatorial equation and gives explicit asymptotic formulas, determined directly from the corresponding functional equations. The principal results are summarized in Theorem 1. Two applications of the main theorem are detailed: Asymptotic enumeration of partition lattice chains and Takeuchi numbers. This technique is equally amenable to the asymptotic enumeration of Bell numbers.

We begin with brief descriptions of these two problems. For each example we give a functional equation satisfied by a generating function of the family.

1.1. Partition lattice chains. The set of partitions of an n -set can be ordered by subset inclusion to build a poset. Define Z_n as the number of chains from the minimal element $\{\{1\}, \{2\}, \dots, \{n\}\}$ to the maximal element $\{1, 2, \dots, n\}$. This sequence begins $Z_1 = 1, Z_2 = 2, Z_3 = 4, Z_4 = 32$. These numbers satisfy the following recurrence, due to Lengyel [4]:

$$Z_n = \sum_{k=1}^{n-1} S_{n,k} Z_k$$

where the $S_{n,k}$ are the Stirling numbers of the second kind. From this, we deduce the functional equation for the exponential generating function $Z(z) = \sum_n Z_n \frac{z^n}{n!}$, also due to Lengyel:

$$(1) \quad Z(z) = \frac{1}{2}Z(e^z - 1) + \frac{z}{2}.$$

In the final section we give an asymptotic formula for Z_n , which matches previous work by Flajolet and Salvy.

1.2. Takeuchi Numbers. Consider the following recursive function of Takeuchi, related to ballot numbers:

$$\text{TAK}(x, y, z) := \text{if } x \leq y \text{ then } y \text{ else } \text{TAK}(\text{TAK}(x - 1, y, z), \text{TAK}(y - 1, z, x), \text{TAK}(z - 1, x, y)).$$

Denote by $T(x, y, z)$ number of times the **else** clause is invoked when evaluating $\text{TAK}(x, y, z)$. Define the sequence T_n by $T_n = T(n, 0, n+1)$. The initial terms are $T_1 = 1, T_2 = 4, T_3 = 14, T_4 = 53$.

Knuth determined the following recurrence [3], and its corresponding functional equation for the ordinary generating function $T(z) = \sum_n T_n z^n$:

$$T_{n+1} = \sum_{k=0}^n \left[\binom{n+k}{n} - \binom{n+k}{n+1} \right] T_{n-k} + \sum_{k=1}^{n+1} \binom{2k}{k} \frac{1}{k+1};$$

$$(2) \quad T(z) = zC(z)T(zC(z)) + \frac{C(z)-1}{1-z}, \quad C(z) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{z^k}{k+1}.$$

The methodology presented here yields a new result for the asymptotic expansion of T_n .

1.3. General setup. The common feature of these two problems is that they satisfy a linear recurrence of the form

$$X_n = \sum_{k=1}^n c_{n,k} X_{n-k} + b_n,$$

with a functional equation for either the ordinary or exponential generating function $X(z)$ of the form:

$$X(z) = a(z)X \circ f(z) + b(z),$$

where $f(z) = z + cz^2 + dz^3 + \dots$ has a parabolic fixed point. This is the functional equation associated with the following combinatorial equation where \circ denotes the substitution operation: $\mathcal{X} = \mathcal{A} \times \mathcal{X} \circ \mathcal{F} + \mathcal{B}$. The remainder of this work is devoted to determining an asymptotic expression for X_n .

2. Asymptotic Analysis

The asymptotic analysis X_n has three major steps. First, we determine an expression for X_n as an integral, and then we perform a two step analysis on this integral, first using analytic iteration theory and then using a saddle-point analysis.

2.1. An expression for the coefficient. If a formal power series satisfies Eq. (1.3), with $a(z)$, $f(z)$, and $b(z)$ analytic near $z = 0$, then we have the formal solution

$$X(z) = \sum_{m=0}^{\infty} \left(\prod_{k=0}^{m-1} a \circ f^k(z) \right) b \circ f^m(z).$$

We use this formal solution and the Cauchy inversion formula to determine an expression for the coefficients of the generating series. We have that $X_n = \sum_{m=0}^{\infty} X_{n,m}$ with

$$(3) \quad X_{n,m} = \frac{1}{2\pi i} \oint \left(\prod_{k=0}^{m-1} a \circ f^k(z) \right) b \circ f^m(z) \frac{dz}{z^{n+1}}.$$

2.2. Analytic iteration theory. To illustrate the general idea, we consider a slightly simpler problem. Let $Y(z)$ be a solution of the homogeneous equation

$$(4) \quad Y(z) = a(z) Y \circ f(z).$$

In this case we have

$$\prod_{k=0}^{m-1} a \circ f^k(z) = \frac{Y(z)}{Y \circ f^m(z)}.$$

With this, Eq. (3) rewrites as

$$(5) \quad X_{n,m} = \frac{1}{2\pi i} \oint \frac{Y(z)}{Y \circ f^m(z)} b \circ f^m(z) \frac{dz}{z^{n+1}} = \frac{1}{2\pi i} \oint \frac{b \circ f^m(z)}{Y \circ f^m(z)} Y(z) \frac{dz}{z^{n+1}}.$$

To establish the existence of $Y(z)$ and certain analyticity properties, we use analytic iteration theory, see [1, 5], and some astute observations.

First, we use the parabolic linearization theorem to show the conjugacy of $f(z)$ to a shift. We have that $f^{-1}(z)$ exists in some cardioid domain and maps contractively to it, via some (determinable) function μ . We deduce that $f^k(z) = \mu(\mu^{-1}(z) - k)$ for z sufficiently small. Given a complete asymptotic expansion for μ , we have that $f^{-m} \circ \mu(s) = \mu(s + m)$ admits a complete asymptotic expansion for $m \rightarrow \infty$ of the form:

$$(6) \quad \mu(s + m) \sim \frac{1}{cm} \left(1 + \left(1 - \frac{d}{c^2} - s \right) \frac{\log m}{m} + \sum_{k=2}^{\infty} \sum_{j=0}^k \nu_{j,k}(s) \frac{(\log m)^j}{m^k} \right).$$

Substitute $z = \mu(s)$ into Eq (4), and capitalize on the resulting similarities to the gamma function to determine a solution to the homogeneous equation. Most importantly, we deduce the following asymptotic result:

$$(7) \quad \frac{Y \circ \mu(s + n)}{Y \circ \mu(n)} \sim (a \circ \mu(n))^s.$$

Next, substitute $z = \mu(s + m)$ and Eq. (7) into the last integral in Eq. (5) and then apply the asymptotic expansion of $\mu(s + m)$ from Eq. (6) to get the following asymptotic formula:

$$X_{n,m} \sim (cm)^n m^{-1 - (1 - \frac{d}{c^2}) \frac{n}{m}} \frac{Y \circ \mu(m)}{2\pi i} \int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} \left(a \circ \mu(m) e^{\frac{n}{m}} \right)^s ds.$$

Returning to X_n , we see that the sum simplifies to

$$X_n \sim C \sum_m (cm)^n \frac{Y \circ \mu(m)}{m} (a \circ \mu(m))^{(1 - \frac{d}{c^2}) \log m}$$

with

$$(8) \quad C = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} ds.$$

2.3. Saddle-point analysis. We conclude by applying a saddle-point analysis to the integral in Eq. (8). There is a saddle point at $a \circ \mu(m) e^{\frac{n}{m}} = 1$. The saddle has different behavior depending on the smallest term of $a(z) = a_k z^k + \dots$. We can summarize this analysis in the following theorem, which treats the $k = 0, 1$ cases separately.

Theorem 1. Suppose the formal power series $X(z) = \sum_{n=0}^{\infty} X_n z^n$ satisfies

$$X(z) = a(z)X \circ f(z) + b(z)$$

with $f(z) = z + cz^2 + dz^3 + \dots$, $a(z) = a_k z^k + \dots$, and $b(z)$ analytic near zero. If $c > 0$ and $0 < a_k$ then the following are true:

1. If $k = 0$, and $a_0 < 1$, then

$$X_n \sim Dn! (-c/\log a_0)^n n^{(1-\frac{d}{c^2})\log a_0 - 1} \quad \text{as } n \rightarrow \infty, \text{ where } D = (-\log a_0)^{-(1-\frac{d}{c^2})\log a_0};$$

2. If $k = 1$ then

$$X_n \sim Dc^n e^{-\frac{1}{2}(1-\frac{d}{c^2})W(\frac{c}{a_1}n)^2} \sum_{m=0}^{\infty} \frac{m^n}{m!} \left(\frac{a_1}{c}\right)^m \quad \text{as } n \rightarrow \infty, \text{ where } D = e^{\frac{1}{2}(1-\frac{d}{c^2})(\log \frac{a_1}{c})^2}.$$

3. Combinatorial Applications

We now possess sufficiently many tools to determine some asymptotic results with our earlier examples.

3.1. Partition lattice chains. Using Eq. (1) we deduce $a(z) = \frac{1}{2}$, $f(z) = e^z - 1$, $b(z) = \frac{z}{2}$. We have $\mu(s) \sim \frac{2}{s}(1 - \frac{\log s}{3s} + \dots)$, and $Y \circ \mu(s) = 2^s$ thus we insert $c = \frac{1}{2}$, $d = \frac{1}{6}$, $a_0 = \frac{1}{2}$ into the main theorem, part 1. The resulting asymptotic expansion is

$$Z_n \sim D(n!)^2 (2 \log 2)^{-n} n^{-1-\frac{1}{3} \log 2}$$

as $n \rightarrow \infty$, where

$$D = \frac{1}{2} (\log 2)^{\frac{1}{3} \log 2} \frac{1}{2\pi i} \int_{\mathcal{C}} 2^s \mu(s) ds = 1.0986858055 \dots$$

3.2. Takeuchi numbers. From Eq. (2), we have $a(z) = zC(z)$, $f(z) = zC(z)$, and $b(z) = \frac{C(z)-1}{1-z}$. From this we determine $\mu(s) \sim \frac{1}{s}(1 - \frac{\log s}{s} + \dots)$, and thus, $Y \circ \mu(s) \sim e^{-\frac{1}{2}(\log s)^2} / \Gamma(s)$. Denote by B_n the n th Bell numbers. Applying these values to part 2 of the main theorem yields the asymptotic expansion:

$$T_n \sim D \sum_{m=0}^{\infty} \frac{m^n}{m!} e^{\frac{1}{2}W(n)^2} = D' B_n e^{\frac{1}{2}W(n)^2}$$

as $n \rightarrow \infty$, where

$$D' = \frac{e}{2\pi i} \int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} ds = 2.2394331040 \dots$$

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