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Analytic Urns

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Abstract

The talk[1] describes an analytic approach to urn models of the Pólya type where an urn may contain balls of either of two colours. At each step, a ball is randomly drawn and replaced by balls of the two colours; a fixed 2×2-matrix with constant row sum determines the replacement policy. The treatment starts from a partial differential equation associated with the model and bases itself on conformal mapping arguments coupled with singularity analysis techniques. This gives access to moments characterizations, Gaussian limits and large deviation results. In some specific and well-determined cases, the urn models admit explicit representations in terms of Weierstraß elliptic functions.

1. Introduction

We follow in this summary the following route:

1. analyze the urns system

$$\mathcal{T}_{2,3} = \begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix}, \tag{1}$$

   that models also the 2-3 trees, and leads to explicit results involving elliptic functions;
2. briefly sketch the analysis done by Flajolet et al. in the general case.

For complete proofs and further references, see the article "Analytic urns" of Philippe Flajolet, Joaquim Gabarró, and Helmut Pekari [1].

2. Analysis of the $\mathcal{T}_{2,3}$ Model

In the $\mathcal{T}_{2,3}$ model of Equation 1, when drawing a black ball one pulls out 2 black balls and adds 3 white balls, while when drawing a white ball one pulls out 3 white balls and adds 4 black balls; in the equivalent $\mathcal{T}_{2,3}$ tree model, black (resp. white) balls count the number of keys in 2-nodes (resp. 3-nodes), and the system models the transformation of a 2-node into a 3-node and the splitting of a 3-node into two 2-nodes. The initial counts are 2 black balls and 0 white balls, that insures a *tenable* system that cannot be blocked by removing more balls of any color than present in the urn.

[1]This talk presents a joint work of Philippe Flajolet with Joaquim Gabarró, and Helmut Pekari of University of Barcelona.

2.1. **The basic PDE.** Let X_n be the random variable representing the number of black balls at time n. The dynamics of the process is expressed by the stochastic recurrence

$$X_1 = 2; \qquad X_n - X_{n-1} = \begin{cases} -2 & \text{with probability } \dfrac{X_{n-1}}{n} \\ +4 & \text{with probability } 1 - \dfrac{X_{n-1}}{n}. \end{cases} \tag{2}$$

Let $p_{n,k} = \mathbf{P}(X_n = k)$. We also define

$$p_n(u) := \sum_k p_{n,k} u^k = \mathbf{E}(u^{X_n}) \quad \text{and} \quad F(z,u) := \sum_{n\geq 1} p_n(u) z^n = \sum_{n,k} p_{n,k} u^k z^n.$$

where $p_n(u)$ is the *probability generating function* (PGF) of X_n.

The recurrence 2 translates into a recurrence on the $p_{n,k}$, and the PGF $p_n(u)$ satisfies the differential recurrence,

$$p_n(u) = u^4 p_{n-1}(u) + \frac{1}{nu}(1-u^6)\frac{d}{du}p_{n-1}(u), \tag{3}$$

for $n \geq 2$, together with the initial condition $p_1(u) = u^2$. From this, we get a *partial differential equation* (PDE) satisfied by the bivariate generating function (BGF) $F(z,u)$:

$$(u^5 z - u)\frac{\partial F}{\partial z} + (1-u^6)\frac{\partial F}{\partial u} + u^5 F + u^3 = 0. \tag{4}$$

A slightly modified version of this is

$$G(z,u) := p_0(u) + F(z,u), \quad \text{with} \quad p_0(u) = (1-u^6)^{1/6}\int_0^u t^3(1-t^6)^{-7/6}\,dt \tag{5}$$

given by solving Equation 3 for $n = 1$. The function G is now a solution of the homogeneous equation

$$(u^5 z - u)\frac{\partial G}{\partial z} + (1-u^6)\frac{\partial G}{\partial u} + u^5 G = 0. \tag{6}$$

2.2. **The solution by quadrature.** The algorithm given in Figure 1 provides a general way to solve quasi-linear partial differential equations of the first order. Applying this method to the PDE (6), we aim at solving the ordinary differential system:

$$\frac{du}{1-u^6} = \frac{dz}{u^5 z - u} = -\frac{dw}{u^5 w}. \tag{14}$$

the solutions $w = w(u)$ and $z = z(u)$ respectively verifies

$$w(1-u^6)^{-1/6} = C_1 = U(z,u,w) \quad \text{and} \quad z(1-u^6)^{1/6} + \int_0^u \frac{t}{(1-t^6)^{5/6}}\,dt = C_2 = V(z,u,w). \tag{15}$$

for some arbitrary integration constant C_1 and C_2. (Use the variation-of-constant method to compute $z(u)$). With

$$I(u) := \int_0^u \frac{t}{(1-t^6)^{5/6}}\,dt, \qquad \delta(u) := (1-u^6)^{1/6}.$$

we obtain the functional equation (with the notations of Figure 1 and $w = G$)

$$\Phi\left(\frac{G}{\delta(u)}, z\delta(u) + I(u)\right) = 0,$$

where Φ is an arbitrary function.

Start with a *quasi-linear* (bivariate) partial differential equation of the form

$$A(z,u,G)\frac{\partial G(z,u)}{\partial z} + B(z,u,G)\frac{\partial G(z,u)}{\partial u} + C(z,u,G) = 0, \tag{7}$$

where A, B, C are given functions.

1. First look for a solution in implicit form $X(z,u,G) = 0$. A calculation shows that the trivariate X must satisfy the *linear* (trivariate) partial differential equation:

$$A(z,u,w)\frac{\partial X(z,u,w)}{\partial z} + B(z,u,w)\frac{\partial X(z,u,w)}{\partial u} - C(z,u,w)\frac{\partial X(z,u,w)}{\partial w} = 0. \tag{8}$$

2. Next consider the ordinary differential system

$$\frac{dz}{A} = \frac{du}{B} = -\frac{dw}{C}. \tag{9}$$

The solution of two "independent" ordinary differential equations induced by (9), e.g.,

$$\frac{du}{B} = -\frac{dw}{C} \quad \text{and} \quad \frac{dz}{A} = \frac{du}{B}, \tag{10}$$

leads to two families of integral curves known as "first integrals,"

$$U(z,u,w) = C_1 \quad \text{and} \quad V(z,u,w) = C_2, \tag{11}$$

with z and w respectively treated as parameters. Assuming nondegeneracy, the generic solution of the PDE (8) is provided by

$$X(z,u,w) = \Phi(U(z,u,w), V(z,u,w)), \tag{12}$$

for an arbitrary bivariate Φ.

3. The trivariate X determines G implicitly by $X(z,u,G) = 0$, that is, by (12) one must have $\Phi(U(z,u,G), V(z,u,G)) = 0$. Solving for G provides a relation $G = R_\Phi(z,u)$, where R_Φ depends upon the arbitrary function Φ. The general solution of (7) is then

$$G(z,u) := R_\Phi(z,u). \tag{13}$$

FIGURE 1. The solution algorithm for quasilinear PDEs of first order.

Solving for G gives

$$G(z,u) = \delta(u)\Psi(\delta(u)z + I(u)), \tag{16}$$

where Ψ is an *arbitrary* function.

The unknown function Ψ is identified from the boundary condition,

$$G(0,u) = p_0(u), \tag{17}$$

that implies, assuming that (16) remains valid in this boundary case:

$$\frac{p_0(u)}{\delta(u)} = \Psi(I(u)) \quad \text{or} \quad J(u) := \int_0^u t^3(1-t^6)^{-7/6}\,dt = \Psi(I(u)). \tag{18}$$

The relation (18) then provides a *parameterization* of Ψ, hence it eventually determines a plausible value for G. This gives (with the previous notations):

Theorem 1. *The bivariate generating function of the probabilities is*

$$G(z,u) = \delta(u)\Psi\left(z\delta(u) + I(u)\right), \tag{19}$$

where Ψ is the function defined parametrically for $|u| < 1$ by

$$\Psi(I(u)) = J(u). \tag{20}$$

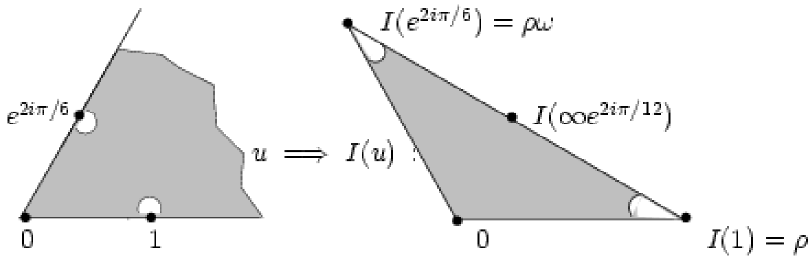

FIGURE 2. The "elementary triangle" T_0 (right) is the image of the basic sector S_0 (left) via the mapping $u \mapsto I(u)$.

2.3. **Dominant singularity.** The map $u \mapsto I(u)$ is analytic in the unit disk $|u| < 1$. Since $I(u)$ has nonnegative Taylor coefficients at 0, the image of the unit dis in the u-plane is a subset of the Dis centered at the origin and having radius ρ defined by $\rho = I(1)$. In particular, as $u \to 1^-$, one has $z \to \rho^-$. On the other hand the integral $J(u)$ diverges to $+\infty$ as $u \to 1^-$. There results that $z = \rho$ is a singularity of $\Psi(z)$. The quantity ρ happens to admit of closed form in terms of Gamma function factors (a complete Eulerian beta integral), and using Pringsheim's theorem, we get:

Lemma 1. *The function $\Psi(z)$ is analytic in the disco $|z| < \rho$, where*

$$\rho \equiv I(1) = \frac{1}{6}B\left(\frac{1}{6}, \frac{1}{3}\right) = \frac{1}{6}\frac{\Gamma(1/3)\Gamma(1/6)}{\Gamma(1/2)} \doteq 1.40218\,21053\,25454. \tag{21}$$

2.4. **The fundamental triangle.** An expansion of Ψ near 0 exhibits a periodicity of the coefficients of Ψ modulo 3. Local expansions of $I(u)$ and $J(u)$ show the limiting behaviors (in the sense of directional limits):

$$\Psi(z) \underset{z\to\rho}{\sim} \frac{1}{\rho - z}, \qquad \Psi(z) \underset{z\to\rho\omega}{\sim} \frac{1}{\rho\omega - z}, \qquad \Psi(z) \underset{z\to\rho\omega^2}{\sim} \frac{1}{\rho\omega^2 - z}.$$

Thus Ψ must be singular at the three points ρ, $\rho\omega$, and $\rho\omega^2$.

We consider $\delta(u) = (1 - u^6)^{1/6}$. Let $\zeta = e^{2i\pi/6}$ and define the region R_0 as the complex plane slit along the six rays $\zeta^j t$ with $t \in [1, +\infty[$, where $j = 0, \dots, 5$. Let $R_0, \dots, R_5$ be six copies of R_0 where by convention $\delta(t) \sim \zeta^j$ when $t \sim 0$ in R_j, with δ being also extended by continuity on R_j. By conveniently "gluing" together the six copies $R_0, \dots, R_5$ along the rays, one obtain the Riemann Surface of δ onto which δ is single-valued.

We can restrict ourself by symmetry to the "slit" half-plane $\mathcal{H}$, where

$$\mathcal{H} := \{\, z : \Im(z) > 0 \vee (\Im(z) = 0 \wedge \Re(z) \geq 0) \,\}. \tag{22}$$

We then have:

Lemma 2. *The function $I(u)$ maps the interior of $(R_0 \cap \mathcal{H})$ conformally (i.e., in a one-to-one analytic manner) onto the interior of the equilateral triangle T with vertices $\rho, \rho\omega, \rho\omega^2$, where $\omega := e^{2i\pi/3}$.*

Figures 2 and 3 give hints of the proof that is omitted.

2.5. **Analytic continuation and elliptic connection.** The function Ψ is amenable to analytic continuation beyond its disk of convergence $|z| < \rho$. This can be done by rotating the fundamental triangle around $\rho, \rho\omega$ and $\rho\omega^2$. This is further extended to the whole complex plane punctured by the hexagonal lattice

$$\Lambda_{\text{hex}} := \left\{ n_1 e^{i\pi/6} + n_2 e^{i\pi/6} : n_1, n_2 \in \mathbb{Z} \right\}$$

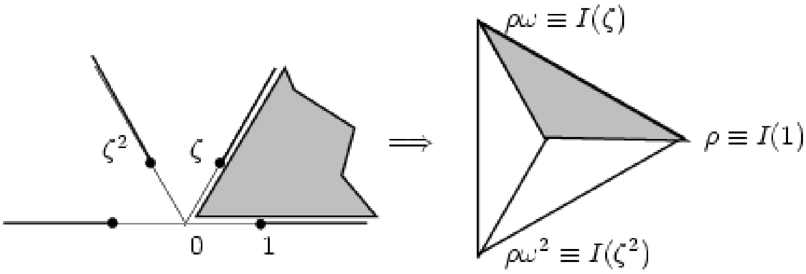

FIGURE 3. The "fundamental triangle" T (right) is the image of the slit upper half-plane $(R_0 \cap \mathcal{H})$ (left) via the mapping $u \mapsto I(u)$.

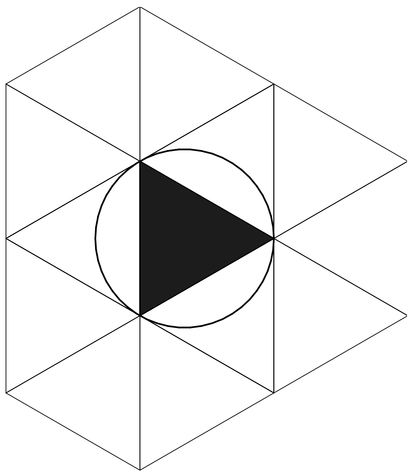

FIGURE 4. Rotated copies of the fundamental triangle around $\rho, \rho\omega, \rho\omega^2$ shown against the circle of convergence of $\Psi(z)$.

There we have:

Theorem 2. *The Ψ-function of the $\mathcal{T}_{2,3}$ model initialized with 2 balls of the first type is exactly*

$$\Psi(z) = \frac{1}{\rho\sqrt{3}}\left(-\zeta\left(\frac{z-\rho}{\rho\sqrt{3}}\right) + \zeta\left(-\frac{1}{\sqrt{3}}\right)\right), \qquad \textit{with} \quad \rho := \frac{1}{6}\,\frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{6})}{\Gamma(\frac{1}{2})}, \tag{23}$$

where $\zeta(z) := \zeta(z;\Lambda_{\text{hex}})$ is the Weierstraß zeta function of the hexagonal lattice defined by

$$\zeta(z;\Lambda_{\text{hex}}) := \frac{1}{z} + \sum_{w\in\Lambda_{\text{hex}}\setminus\{0\}}\left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2}\right), \tag{24}$$

Sketch of proof: (1) each point z of the punctured points is reachable by a path $I(\gamma(u))$ for suitable γ; (2) poles and residues of both functions are the same; (3) Liouville theorem.

2.6. **Probabilistic consequences.** A consequence of the results of the preceding section is:

Corollary 1. *For the $\mathcal{T}_{2,3}$ model, the probability generating function $p_n(u) = \mathbf{E}(u^{X_n})$ admits an* exact *formula valid for all $n \geq 2$,*

$$p_n(u) = \sum_{n_1,n_2=-\infty}^{+\infty}\left(K(u) + \frac{\rho\sqrt{3}}{\delta(u)}(n_1 e^{i\pi/6} + n_2 e^{-i\pi/6})\right)^{-n-1}, \tag{25}$$

where

$$K(u) := \frac{1}{\delta(u)} \int_u^1 \frac{t}{\delta(t)^5}\, dt, \qquad \delta(u) = (1-u^6)^{1/6}.$$

From this,(1) application of the Quasi-Powers Theorem of Bender and Hwang provides a Gaussian limit law, so as speed of convergence to the limit; (2) there are exact polynomial forms for the moments; (3) a large deviation law with exponential decay follows.

3. General Case

In the general case, the urn model with replacement is specified by the matrix

$$\begin{pmatrix} -a & a+s \\ b+s & -b \end{pmatrix},$$

with initial conditions: $a = a_0$ (black), $b = b_0$ (white).

The combinatorial analysis uses a "history" model with stamped urns; this leads to a PDE for the bivariate generating function $F(z,u)$ counting time and black balls, via a differential operator.

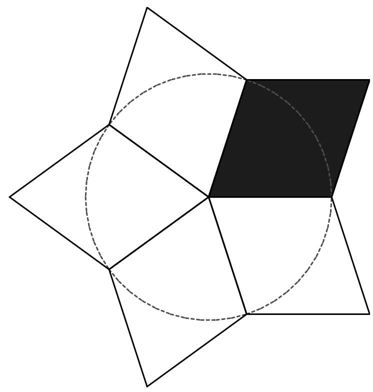

FIGURE 5. The fundamental polygon associated with the urn $(-1, 4, 4, -1)$.

The main features of the analysis are:

- For each system, there exists a regular fundamental polygon inside which the function Ψ is analytic.
- There are isolated singularities of Puiseux type on the boundary of the disk of convergence of $F(z,u)$ where u is considered as a parameter. This entails asymptotic equivalents for $[z^n]F(z,u) = p_n(u)$.
- The Quasi-Powers Theorem applies to $p_n(u)$, implying a Gaussian limit.
- The rth factorial moment of X_n is of hypergeometric type.
- The large deviation rate is fully characterized.

In general, the Puiseux singularities preclude solutions in terms of elliptic functions. This corresponds also to the impossibility of tiling the complex plan with the fundamental polygon (see for instance Figure 5 where $\Psi \asymp (\rho - z)^{-1/3}$).

The six elliptic urns are:

$$\begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 3 \\ 5 & -3 \end{pmatrix}, \quad \begin{pmatrix} -1 & 4 \\ 5 & -2 \end{pmatrix}.$$

Bibliography

[1] Flajolet (P.), Gabarró (J.), and Pekari (H.). – Analytic urns. – Submitted to *Annals of Probability*, available at `http://algo.inria.fr/flajolet/Publications/`.