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Queues, Stacks, and Transcendentality at the Transition to Chaos

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Summary by Bruno Salvy

Iteration of the logistic map

$$F_\mu(x) = 4\mu x(1-x), \qquad \mu \in [0,1)$$

is a classical example of a discrete dynamical system exhibiting chaos. Depending on the value of μ, the iterates of an arbitrary $x \in I = [0,1]$ are attracted to a limit cycle of size a power of 2 (see [3]). Figure 1 displays the values of $F_\mu^{50}(1/2), \ldots, F_\mu^{100}(1/2)$ as μ increases from 0 to 1, where F^k denotes the kth iterate of F. Figure 2 shows an example of a trajectory with an attracting 4-cycle.

To each $x \in I$ is associated the infinite word $a(x) \in \{0,1\}^\star$ whose kth letter is 0 if $F_\mu^k(x) \le 1/2$ and 1 otherwise. The aim of Cristopher Moore and Porus Lakdawala [6] is to study the language L formed by the set of prefixes of all $a(x)$ for $x \in I$ (the *symbolic dynamics* of F_μ) and its evolution as μ increases from 0 to 1. For instance, the language corresponding to μ in Figure 2 is

$$L = 0^\star 1^\star (10)^\star (1011)^\star.$$

This can be interpreted as follows: the first iterates can be smaller than $1/2$, but apart from the fixed point at 0 (where $a(0) = 0^\star$) they eventually get larger. Then, apart from the second fixed point of F_μ (where a is $1^\star$) the iterates are attracted by the 4-cycle, but they may first have a few iterates on the other side of $1/2$, hence the $(10)^\star$. One should also account for those prefixes which

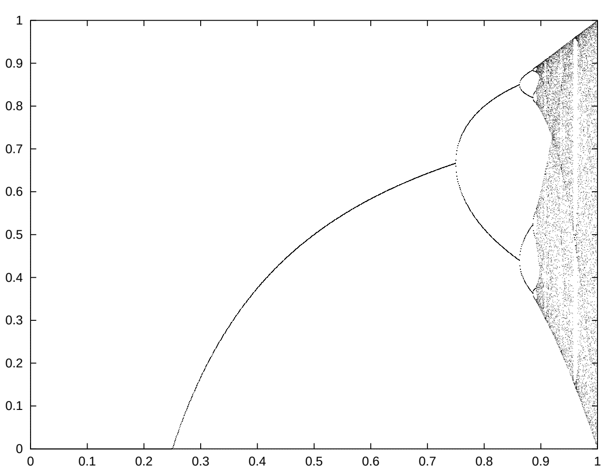

FIGURE 1. The period-doubling phenomenon.

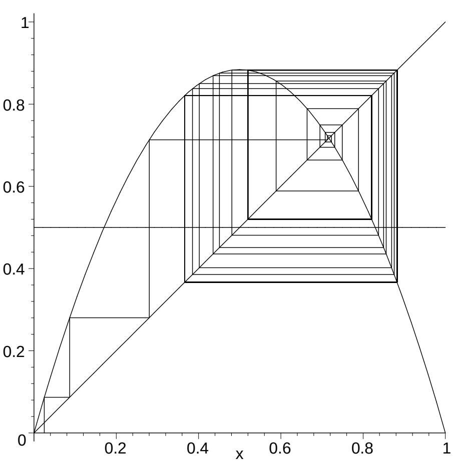

FIGURE 2. 100 iterates for $\mu = 0.884$.

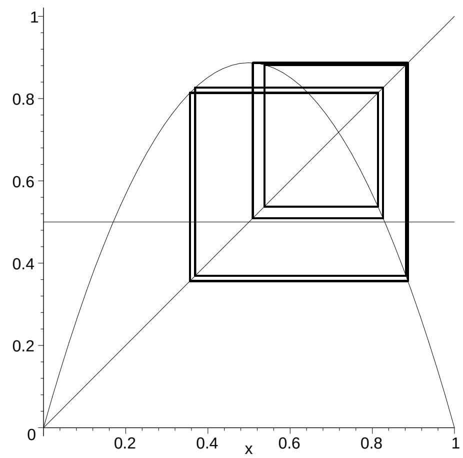

FIGURE 3. Limit cycle for $\mu = 0.887$.

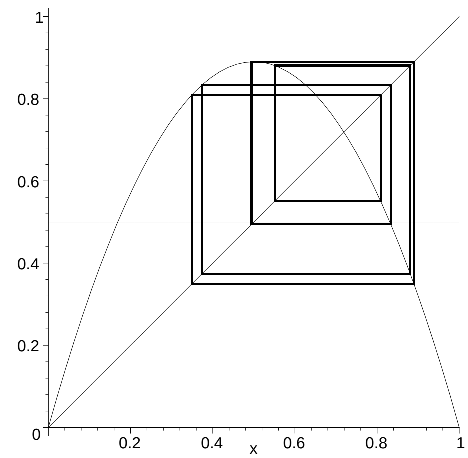

FIGURE 4. Limit cycle for $\mu = 0.89$.

do not end exactly at the end of a period; this is obtained by concatenating $(\epsilon|1|10|101)$ at the end of L and removing $(1|10)$ which otherwise would be counted twice. However, these modifications introduce unnecessary technicalities and will be ignored in what follows. When μ increases further, the 4-cycle becomes repelling and gives rise to an attracting 8-cycle. This does not change L until the third element of the cycle becomes smaller than 1/2, and then

$$L = 0^\star 1^\star (10)^\star (1011)^\star (10111010)^\star.$$

Examples of corresponding 8-cycles are given in Figure 3 and 4.

1. **Transcendentality at the Transition to Chaos**

This process leads to a sequence of languages

$$L_0 = 0^\star, \quad L_1 = L_0 w_0^\star, \quad L_2 = L_1 w_1^\star, \dots, \tag{1}$$

with $w_0 = 1$ and $w_{n+1} = R(w_n)$ where R is the substitution

$$R : 0 \mapsto 11, 1 \mapsto 10. \tag{2}$$

Each of these languages is regular. Their generating functions are obtained by translation from (1):

$$L_0(z) = \frac{1}{1-z}, \qquad L_n(z) = L_{n-1}(z)\frac{1}{1-z^{2^n}}.$$

The *transition to chaos* corresponds to letting μ approach 1. The limiting value of w_n is the fixed point of R, the *Morse sequence*. The limiting value of L has a generating function defined by

$$L_\infty(0) = 1, \qquad L_\infty(z) = \frac{L_\infty(z^2)}{1-z}. \tag{3}$$

From this it follows that $L_\infty(z)$ has an infinite number of singularities on the unit circle, thus $L_\infty(z)$ is not algebraic and the corresponding language is not context-free. This generating function is classical: it is the generating function of binary partitions studied by Mahler [5] and de Bruijn [2] who showed that the logarithm of the nth Taylor coefficient of L_∞ behaves asymptotically like

$$\frac{1}{2\log 2}\left(\log\frac{n}{\log n}\right)^2+\left(\frac{1}{2}+\frac{1}{\log 2}+\frac{\log\log 2}{\log 2}\right)\log n \\ -\left(1+\frac{\log\log 2}{\log 2}\right)\log\log n+F\left(\frac{\log n-\log\log n}{\log 2}\right)+o(1),$$

where F is a periodic function with period 1 for which a full Fourier expansion is known.

2. Stacks of Stacks

Since the language L_∞ is not context-free, it cannot be recognized with a finite amount of memory. The question addressed by Moore and Lakdawala is to determine how simple a long-term memory mechanism recognizing L_∞ can be. This in turn is expected to give more precise information on the nature of the transition to chaos. Two natural candidates for the mechanism are the *queue* (first in–first out) and the *stack* (last in–first out).

Since context-free languages are those recognized by automata with a stack (*pushed-down automata*) [4], a stack is not sufficient to recognize L_∞. A more general class of languages is provided by *indexed languages* [4, p. 389], whose grammars look like context-free grammars except for string indices, which can be appended to non-terminals. Production rule involving an indexed non-terminal copies this index to all non-terminals it produces. For instance, $\{\, a^nb^nc^n \mid n\ge 0\,\}$ is not context-free but it is indexed, the grammar being

$$\begin{array}{lll} S\to T_{fg}, & T\to T_f, & T\to ABC,\\ A_f\to aA, & B_f\to bB, & C_f\to cC,\\ A_g\to a, & B_g\to b, & C_g\to c. \end{array}$$

From the start state, the first rule introduces a final g, the second one stacks any number of f's to produce T_{f^ng}. The third rule then produces $A_{f^ng}B_{f^ng}C_{f^ng}$, the rules on the second line pop these indices and the final g is popped by the rules on the third one. More generally, these languages are recognized by *nested stack automata* which resemble stacks of stacks.

It turns out that L_∞ can be recognized by such a grammar:

$$\begin{array}{ll} S\to 0S\mid T, & T\to A_g\mid A_gT\mid T_f,\\ A_f\to AB, & B_f\to AA,\\ A_g\to 1, & B_g\to 0. \end{array}$$

The first rule takes care of the initial $0^\star$, the second one first stacks a number k of f's at the end and then either produces an A_{f^kg} or an $A_{f^kg}T_{f^k}$. To this final T_{f^k}, more f's can then be stacked by that same rule. To see that L_∞ is the end result, it is then sufficient to show why A_{f^kg} actually produces the word w_k from (1). This follows from productions in the second line performing the substitution R from (2).

3. Queues

Automata with k queues can simulate the k tapes of a multi-tape Turing machine. However, restricting the way the queues are accessed by imposing a bound on the number of transitions performed for each symbol of the input string leads to the class of *quasi-real-time queue automata* [1]. The corresponding grammars are *breadth-first grammars.* In these grammars, a production of the form $A\to sB$ where s is a string of terminals and B a string of non-terminals rewrites a string xAy into $xsyB$ and the rule has to be applied to the leftmost non-terminals first. Thus the string of non-terminals represents the queue and the string of terminals represents the part of the input that has been read so far.

By storing the current w_n on the queue and applying R when necessary to expand it, Moore and Lakdawala show that L_∞ is recognizable by a real-time deterministic queue automaton with one queue.

4. Stacks

Again, with no time restriction, two stacks are sufficient to simulate a universal Turing machine. Exploiting the fact that w_n is a palindrome except for its last symbol, it can be shown [6] that L_∞ can be recognized by a real time automaton with two stacks.

The conclusion [6] is therefore that since one queue is sufficient while two stacks are necessary, the long-term memory of the system has more of a FIFO character. It is unclear however how much of this work can be generalized to other dynamical phase transitions.

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