Some Sharp Concentration Results about Random Planar Triangulations

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Submap Density Result

Let T_0 be any fixed triangulation, and $\eta_n(T_0)$ be the number of copies of T_0 in a random triangulation with n vertices.

Richmond and Wormald (1988) :

$$\mathbf{P}(\eta_n(T_0) > cn) > 1 - e^{-\delta n}$$

for some positive constants c and δ. (depending on T_0)

Bender, Gao and Richmond (1992) : The above result holds for many families of maps.

Gao and Wormald : $\eta_n(T_0)$ is sharply concentrated around cn for some constant c.

Theorem 1 *Let T_0 be a 3-connected triangulation with $j+3$ vertices such that there are r distinct ways to root T_0. Let*

$$c = 2r \left(\frac{27}{256}\right)^j .$$

Then

$$\mathbf{P}\left(|\eta_n(T_0) - cn| = o(cn)\right) \to 1,$$

provided that $cn \to \infty$.

Define

$$\mu_k = \frac{8(k-2)}{4k^2-1}\left(-\frac{3}{4}\right)^k \binom{-3/2}{k}.$$

Theorem 2 *Let M be a 3-connected near-triangulation with external face of degree k and with j internal vertices such that there are r distinct ways to root the external face. Then, for fixed j, k with $k \geq 4$,*

$$\mathbf{P}\left(\left|\eta_n(M) - r\mu_k\left(\frac{27}{256}\right)^{j-1} n\right| = o(n)\right) \to 1.$$

A sketch of the proof.

First study the number $\zeta_n(k)$ of vertices of degree k in a random triangulation with $n+2$ vertices.

- T_n denotes the number of rooted triangulations with $n+2$ vertices.

- $T_{n,k}$ denotes the number of rooted triangulations with $n+2$ vertices and root vertex of degree k.

- $T_{n,k,l}$ denotes the number of rooted triangulations with $n+2$ vertices, root vertex of degree k and another distinguished vertex of degree l.

Step 1 Use combinatorial argument to show

$$E(\zeta_n(k)) = \frac{6n}{k}\frac{T_{n,k}}{T_n},$$
$$E(\zeta_n(k)(\zeta_n(k)-1)) = \frac{6n}{k}\frac{T_{n,k,k}}{T_n}.$$

Step 2 Obtain functional equations for the generating functions for $T_{n,k,l}$, $T_{n,k}$ and T_n, and perform singularity analysis.

Step 3 Derive a multivariate version of Flajolet and Odlyzko's transfer theorem, and obtain the following asymptotics :

$$
\begin{aligned}
T_n &= (\sqrt{6}/(32\sqrt{\pi})n^{-5/2}(256/27)^n(1+O(1/n)),\\
T_{n,k} &= \frac{k\sqrt{6}}{192\sqrt{\pi}}\mu_k n^{-5/2}(256/27)^n\left(1+O\left(k^{20}/n\right)\right),\\
T_{n,k,k} &= \frac{k\sqrt{6}}{192\sqrt{\pi}}\mu_k^2 n^{-3/2}(256/27)^n\left(1+O\left(k^{20}/n\right)\right),
\end{aligned}
$$

uniformly for $k = O(\log n)$.

Step 4 Derive asymptotics for the first two moments of $\zeta_n(k)$.

$$\mathbf{E}(\zeta_n(k)) = n\mu_k \left(1 + O\left(k^{20}/n\right)\right),$$

$$\mathbf{V}(\zeta_n(k)) = n\mu_k + (n\mu_k)^2 O\left(k^{20}/n\right),$$

uniformly for all $k = O(\log n)$.

It follows from Chebyshev's inequality that

$$\mathbf{P}\left(|\zeta_n(k) - \mu_k n| = o(\mu_k n)\right) \to 1$$

uniformly for all

$$k < (\log n - (1/2)\log\log n)/\log(4/3) - \Omega(n).$$

Lemma 1 *Let T_0 be a 3-connected triangulation with $j+3$ vertices such that $j = o(n)$ and there are r distinct ways to root T_0. Let $\eta_n(T_0)$ be the number of copies of T_0 in a random rooted triangulation with $n+2$ vertices. Then*

$$\mathbf{E}(\eta_n(T_0)) = r\left(\frac{27}{256}\right)^{j-1}\mathbf{E}(\zeta_{n+1-j}(3))(1+o(1)),$$

$$\begin{aligned} &\mathbf{E}(\eta_n(T_0)(\eta_n(T_0)-1)) = r^2\left(\frac{27}{256}\right)^{2j-2} \\ \times\ &\mathbf{E}(\zeta_{n+2-2j}(3)(\zeta_{n+2-2j}(3)-1))(1+o(1)). \end{aligned}$$

Maximum Vertex Degree

Let Δ_n be the maximum vertex degree of a random map in a family of maps of size n.

Devroye, Flajolet, Hurtado, Noy and Steiger showed that, for triangulations of an n-gon,

$$\mathbf{P}\left(|\Delta_n - \log n/\log 2| \le (1+\epsilon)\log\log n/\log 2\right) \to 1$$

Gao and Wormald (to appear in JCT-A) showed

- for triangulations of an n-gon,

  $$\mathbf{P}\left(|\Delta_n - (\log n + \log\log n)/\log 2| \le \Omega(n)\right) \to 1$$

- for 3-connected triangulations of n vertices,

  $$\mathbf{P}\left(|\Delta_n - (\log n - (1/2)\log\log n)/\log(4/3)| \le \Omega(n)\right) \to 1$$

- for all maps of n edges,

  $$\mathbf{P}\left(|\Delta_n - (\log n - (1/2)\log\log n)/\log(6/5)| \le \Omega(n)\right) \to 1$$

Parallel Results about Lattice Walks

Neal Madras recently proved a very nice result about patterns in lattice clusters which is parallel to the submap density results. Let $\mathcal{C}_n$ be a set of lattice clusters of size n, and let P_0 be any fixed pattern. Then there is a positive constant ϵ such that the fraction of clusters that contain less than ϵn copies of P_0 (translations of P_0) is exponentially small.

Madras believes that the number of copies should be sharply concentrated around cn for some positive constant c. (which he calls the law of large numbers)

Other Sharp Concentration Results about Triangulations

An example of diagonal flips and flippable edges. Let ζ_n be the number of flippable edges in a random 2-connected triangulation (3-connected triangulation) of n vertices.

Gao and Wang (to appear in JCT-A): ζ_n is sharply concentrated around $5n/2$ ($9n/4$).

ϵ will denote a small positive constant, ϕ is a constant satisfying $0 < \phi < \pi/2$, and $\mathbf{y} = (y_1, y_2, \ldots, y_d)$. Define

$$
\begin{aligned}
\Delta_x(\epsilon,\phi) &= \{x: \ |x| \le 1+\epsilon,\ x \ne 1,\ |\mathrm{Arg}(x-1)| \ge \phi\}, \\
\Delta_j(\epsilon,\phi) &= \{y_j: \ |y_j| \le 1+\epsilon,\ y_j \ne 1,\ |\mathrm{Arg}(y_j-1)| \ge \phi \\
\mathcal{R}(\epsilon,\phi) &= \{(x,\mathbf{y}): |y_j| < 1, 1 \le j \le d, x \in \Delta_x(\epsilon,\phi)\}.
\end{aligned}
$$

Let $\beta_j > 0$ for $1 \le j \le d$, and α be any real number.

Definition 1. We write

$$f(x, \mathbf{y}) = \tilde{O}\left((1-x)^{-\alpha} \prod_{j=1}^{d} (1-y_j)^{-\beta_j}\right)$$

if there are $\epsilon > 0$ and $0 < \phi < \pi/2$ such that in $\mathcal{R}(\epsilon, \phi)$

(i) $f(x, \mathbf{y})$ is analytic, and

$$f(x, \mathbf{y}) = O\left(|1-x|^{-\alpha} \prod_{j=1}^{d} (1-|y_j|)^{-\beta_j}\right)$$

as $(1-x)(1-y_j)^{-p} \to 0$, for $1 \leq j \leq d$, and some $p \geq 0$.

(ii)

$$f(x, \mathbf{y}) = O\left(|1-x|^{-\alpha'} \prod_{j=1}^{d} (1-|y_j|)^{-q}\right)$$

for some $q \geq 0$ and some real number α'.

Definition 2. We write

$$f(x, \mathbf{y}) \approx c\ (1-x)^{-\alpha} \prod_{j=1}^{d} (1-y_j)^{-\beta_j}$$

if $f(x, \mathbf{y})$ can be expressed as

$$\begin{aligned} f(x, \mathbf{y}) \;&=\; c(\mathbf{y})(1-x)^{-\alpha} \prod_{j=1}^{d} (1-y_j)^{-\beta_j} \\ &\quad + \sum_{j=0}^{d} C_j(x, \mathbf{y}) + E(x, \mathbf{y}) \end{aligned}$$

such that

(i) $C_0(x, \mathbf{y})$ is a polynomial in x, and for $1 \leq j \leq d$, $C_j(x, \mathbf{y})$ is a polynomial in y_j.

(ii)

$$E(x, \mathbf{y}) = \tilde{O}\left((1-x)^{-\alpha'} \prod_{j=1}^{d} (1-y_j)^{-\beta'_j} \right),$$

for some $\alpha' < \alpha$ and $\beta'_j \geq 0$, $1 \leq j \leq n$.

(iii) $c(\mathbf{y}) = c + O\left(\sum_{j=1}^{d} |1 - y_j|\right)$ and is analytic in $\{\mathbf{y} : \ y_j \in \Delta_j(\epsilon, \phi)\}$, and $c(\mathbf{1}) = c \neq 0$.

Lemma 2 *Suppose*

$$f(x, \mathbf{y}) = \tilde{O}\left((1-x)^{-\alpha} \prod_{j=1}^{d} (1-y_j)^{-\beta_j}\right).$$

Then

(i) *as* $n \to \infty$ *and* $1 \leq k_j = O(\log n)$ *(*$j = 1, \ldots, d$*),*

$$\left[x^n \mathbf{y}^{\mathbf{k}}\right] f(x, \mathbf{y}) = O\left(n^{\alpha-1} \prod_{j=1}^{d} k_j^{\beta_j}\right);$$

(ii) *for any* $0 < \epsilon' < 1$ *and all* n, k_j*,*

$$\left[x^n \mathbf{y}^{\mathbf{k}}\right] f(x, \mathbf{y}) = O\left(n^{\alpha-1} \prod_{j=1}^{d} (1-\epsilon')^{-k_j}\right).$$

Lemma 3 *Let $d \geq 1$ and*

$$f(x, \mathbf{y}) \approx c\ (1-x)^{-\alpha} \prod_{j=1}^{d} (1-y_j)^{-\beta_j},$$

where α is neither a negative integer nor 0, and $c \neq 0$. Then as $n \to \infty$ and $k_j = O(\log n)$ ($j = 1, \ldots, d$),

$$\left[x^n \mathbf{y}^{\mathbf{k}}\right] f(x, \mathbf{y}) \;=\; \frac{c}{\Gamma(\alpha)} \prod_{j=1}^{d} \left(k_j^{\beta_j - 1} / \Gamma(\beta_j)\right) n^{\alpha-1} \times \left(1 + O\left(\sum_{j=1}^{d} (1/k_j)\right)\right).$$