Asymptotic Combinatorics and Representations of Infinite Symmetric Groups (a Survey)

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[summary by Philippe Chassaing]

Introduction: Multiplicative Measures

In a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ of a nonnegative integer $n = n(\lambda) = \sum_{i=1}^{N} \lambda_i$, the summands λ_i are in decreasing order, and $r_k(\lambda)$ denotes the multiplicity of k, i.e. $r_k(\lambda) = \#\{j | \lambda_j = k\}$. Let $\mathcal{P}_n$ denote the set of partitions of the integer n, and set $\mathcal{P} = \cup_n \mathcal{P}_n$.

One of the questions addressed by this talk is to scale the associated Young diagram $\varphi_\lambda(t)$ defined on $[0, +\infty]$ by

$$\varphi_\lambda(t) = \sum_{k \geq t} r_k(\lambda)$$

in order to obtain nontrivial limit shapes, for the family of *multiplicative measures* on $\mathcal{P}$. A multiplicative measure on $\mathcal{P}$ is defined by a sequence $(\mathcal{F}_k)_{k\geq 1}$ of generating functions

$$\mathcal{F}_k(x) = \sum_{r \geq 0} s_k(r) x^r,$$

as follows: it defines a measure μ^n on $\mathcal{P}_n$ through

$$\mu^n(\lambda) = Q_n^{-1} \prod_k s_k(r_k(\lambda)) = Q_n^{-1} F(\lambda),$$

in which Q_n is chosen so that μ^n is a probability measure on $\mathcal{P}_n$. Assuming that the series

$$\mathcal{F}(x) = \sum_{n \geq 0} Q_n x^n$$

converges for $x \in [0, x_0)$, a probability measure μ_x is then defined on $\mathcal{P}$ as follows:

$$\mu_x = \frac{\sum_n Q_n x^n \mu^n}{\mathcal{F}(x)}.$$

One derives the following facts easily:

- $\mathcal{F}(x) = \prod_{k\geq 1} \mathcal{F}_k(x^k)$;
- according to the measure μ_x, the r_k's are independent random variables;
- the conditional law of a partition λ, given that $\lambda \in \mathcal{P}_n$, is μ^n.

Though the Plancherel measure does not fall in the class of multiplicative measures, most important ones do belong to it.

1. Examples

- *Uniform statistics on $\mathcal{P}_n$.* Let $\mu^n(\lambda) = p(n)^{-1}$, in which $p(n) = \#\mathcal{P}_n$ is the Euler function, $\mathcal{F}_k(x) = 1/(1-x)$ does not depend on k, and as expected:
$$\mathcal{F}(x) = \prod_{k\geq 1} \frac{1}{1-x^k};$$
- *Uniform statistics on partitions with different summands.* $\mathcal{F}_k(x) = 1 + x$, giving
$$\mu_x(\lambda) = x^{n(\lambda)} \prod_{k=1}^{+\infty} (1+x^k)^{-1};$$
- *Bell's statistics.* Here μ^n can be seen as the law of the projection on $\mathcal{P}_n$ of a random partition of a set with n elements, giving:
$$\mu^n(\lambda) = \frac{1}{\prod_k r_k(\lambda)!(k!)^{r_k(\lambda)}}, \qquad \mathcal{F}_k(x) = e^{y/k!}, \qquad \mathcal{F}(x) = e^{e^x-1};$$
- *Haar's statistics and Poisson-Dirichlet measures.* This example is a family $\mu_{x,\theta}$ of multiplicative measures. The case $\theta = 1$ is the partition structure derived from the cycles of a random permutation. The general case arises in various applications in graph theory, or also in genetics under the name of Ewens sampling formula:
$$\mu_\theta^n(\lambda) = \frac{\theta^{\#\lambda}}{[\theta]^{n(\lambda)} \prod_k r_k(\lambda)! k^{r_k(\lambda)}}, \qquad \mathcal{F}_k(x) = e^{\theta y/k}, \qquad \mathcal{F}(x) = (1-x)^{-\theta},$$
in which $\#\lambda$ stands for the number of summands of λ, and $[\theta]^n = \theta(\theta+1)\cdots(\theta+n-1)$.

2. Limit Shapes and Scaling

For a sequence $a = (a_n)_{n\geq 0}$, let $\tau_a\lambda$ denote the scaled Young diagram $t \mapsto a_{n(\lambda)}\varphi_\lambda(a_{n(\lambda)}t)/n(\lambda)$, and for any measure μ on $\mathcal{P}$, let $\tau_a\mu$ denote the image of μ under τ_a. Ergodicity occurs when there exists a normalizing sequence a, such that $\tau_a\mu^n$ converges weakly to a Dirac mass at a given limit shape. The Plancherel measure, as well as the first cases above, are ergodic, but not the last case, where the weak limit is nondegenerate, and can be expressed in terms of the Poisson-Dirichlet distribution with parameter θ. Among the possible tools, a method analog to the saddle-point method allows to derive convergence of $\tau_a\mu^n$ when $n \to +\infty$ from the convergence of $\tau_a\mu_x$ when $x \to x_0$, the latter convergence being easier to prove owing to the independence of r_k's.

The author also developed on heap problems, roof and entropy, on the Young graph, Kingman problem (partition structures), Ulam problem (length of the longest increasing sequence in a sequence of n numbers) and its connection with the spectrum of Gaussian matrices.

Bibliography

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