Summability of Power Series Solutions of q-Difference Equations

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[summary by Michèle Loday-Richaud]

The $\mathbb{C}$-algebra $\mathbb{C}\{x\}[\sigma_q]$ of (linear analytic) q-difference operators is the algebra of polynomials in σ_q where $\sigma_q x = qx\sigma_q$ and where the coefficients are taken in the algebra $\mathbb{C}\{x\}$ of convergent power series at $x = 0$ in $\mathbb{C}$. The elementary operator σ_q acts on x by multiplication by the number q and we make it act on functions of x by $\sigma_q f(x) = f(qx)$. The theory is very different depending on whether $|q|$ is smaller, equal or greater than 1. We deal here with the case when $|q| > 1$ and, for simplicity, we assume that q is a real number.

Like differential equations, q-difference equations may have divergent power series solutions and the aim is to develop a theory of summability for such series like it has been done by Martinet-Ramis and Écalle for solutions of differential equations. A theory of summability means having a rule to change in a unique well-defined way a series solution into an actual solution.

The similarity with differential equations is very strong. However new concepts had to be developed and new phenomena occur.

1. Jacobi equation

The simplest non trivial example is given by the Theta series

$$\Theta(x) = \sum_{n \geq 0} q^{n(n-1)/2} x^n,$$

solution of the Jacobi q-difference equation

$$(J) \qquad xy(qx) - y(x) = -1.$$

The Θ series can be viewed as an analog of the Euler series

$$\sum_{n \geq 0} (-1)^n n! x^{n+1}$$

solution of the Euler equation

$$x^2 y' + y = x.$$

The function

$$y(x) = q^{-\frac{1}{2}(\log_q x - 1)\log_q x},$$

solution of the homogeneous q-difference equation $xy(qx) - y(x) = 0$, is the analog of the exponential function $\exp(1/x)$, solution of the homogeneous differential equation $x^2 y' + y = 0$ and it plays with respect to (J) a like role. Notice however that the series Θ is more divergent than the series solutions of linear differential equations which are known to be of Gevrey type.

Letting

$$y = zq^{-\frac{1}{2}(\log_q x-1)\log_q x}$$

changes (J) into the equation

$$z(qx) - z(x) = -q^{\frac{1}{2}(\log_q x-1)\log_q x}$$

and, letting then $x = q^t$ and $u(t) = z(x)$, into the equation

$$(\Delta) \qquad u(t+1) - u(t) = -q^{\frac{1}{2}(t-1)t}.$$

This latter equation is a linear difference equation the second member of which has an essential singularity at infinity. However the Fourier method can be used to solve it as follows.

Denote by

$$\mathcal{F}(u(t))(\tau) = \frac{1}{2i\pi}\int_{a-i\infty}^{a+i\infty} u(t)e^{-\tau t}dt \qquad \text{and} \qquad \mathcal{F}^{-1}(\varphi(\tau))(t) = \int_{-\infty+ib}^{+\infty+ib} \varphi(\tau)e^{\tau t}d\tau$$

the Fourier and the inverse Fourier transform. Assume that a solution $u(t)$ of (Δ) is left invariant by successive application of $\mathcal{F}$ and $\mathcal{F}^{-1}$. Using the identity $\mathcal{F}(u(t+1))(\tau) = e^{\tau}\mathcal{F}(u(t))(\tau)$ we get

$$\mathcal{F}(u(t))(\tau) = \frac{1}{\sqrt{2\pi\log q}}\frac{q^{-\frac{1}{2}(\frac{1}{2}+\frac{\tau}{\log q})^2}}{1-e^{\tau}}$$

and then solutions of (Δ) in the form

$$u_\theta(t) = \frac{1}{\sqrt{2\pi\log q}}\int_{-\infty+i\theta_q}^{+\infty+i\theta_q}\frac{q^{-\frac{1}{2}(\frac{1}{2}+\frac{\tau}{\log q})^2}}{1-e^{\tau}}e^{\tau t}d\tau.$$

There correspond the following solutions of (J) defined on all of the Riemann surface of log:

$$y_\theta(x) = \frac{q^{-1/8}}{\sqrt{2\pi\log q}}\int_{d_\theta} q^{-\frac{1}{2}(\log_q\frac{x}{\zeta}-1)\log_q\frac{x}{\zeta}}\frac{1}{\zeta(1-\zeta)}d\zeta$$

the integral being taken on the half line d_θ starting from 0 to infinity with angular direction $\theta = \theta_q\log q$ provided that $\theta \neq 0 \mod 2\pi$. When θ varies between two successive forbidden values $2k\pi$ and $2(k+1)\pi$ the corresponding $y_\theta(x)$ are equal. When θ is taken in different such intervals they are equal up to a multiplicative q-constant (a q-constant is a constant in the algebra $\mathbb{C}\{x\}[\sigma_q]$, i.e., a function $C(x)$ satisfying $C(qx) = C(x)$). Thus we can concentrate on one of them. We choose $\theta \in]0, 2\pi[$ and denote by f_0 the corresponding y_θ solution. Such a solution can be taken as a model for q-sums of q-Borel-Laplace summable series.

We emphasize its main property. Writing, for all $\xi \neq 1$, the identity $1/(1-\xi) = \sum_{m=0}^{n-1}\xi^m + \xi^n/(1-\xi)$ yields the equality

$$f_0(x) = \sum_{m=0}^{n-1} q^{m(m-1)/2}x^m + \frac{q^{-1/8}}{\sqrt{2\pi\log q}}\int_{d_\theta} q^{-\frac{1}{2}(\log_q\frac{x}{\xi}-1)\log_q\frac{x}{\xi}}\frac{\xi^{n-1}}{\xi(1-\xi)}d\xi$$

and then the inequality

$$\left| f_0(x) - \sum_{m=0}^{n-1} q^{m(m-1)/2}x^m \right| \leq C_\theta q^{\frac{n(n-1)}{2}+\frac{1}{2}\arg_q^2(xe^{-i\theta})}|x|^n$$

where C_θ is the constant $C_\theta = \max(1, 1/|\sin\theta|)$ and $\arg_q = \frac{1}{\log q}\arg$. Note that the constant C_θ is locally uniform in θ. Such a condition can be taken as a model for f_0 to be the q-sum of level 1 of its Taylor series $\sum_{m\geq 0} q^{m(m-1)/2}x^m$.

We will see that, in all generality, q-Borel-Laplace summable series and q-summable series of level 1 are the same series.

2. q-Borel-Laplace summability or q-summability of level 1

Translating the Fourier and inverse Fourier transforms in terms of the variables $x = q^t$ and $\xi = q^\tau$ yields the q-Borel and q-Laplace transforms

$$\mathcal{B}q(f)(\xi) = \frac{-iq^{1/8}}{\sqrt{2\pi \log q}} \int_{|x|=\rho} q^{\frac{1}{2}(\log_q \frac{x}{\xi} - 1)\log_q \frac{x}{\xi}} f(x) \frac{dx}{x},$$
$$\mathcal{L}q^\theta(\varphi)(x) = \frac{q^{-1/8}}{\sqrt{2\pi \log q}} \int_{d_\theta} q^{-\frac{1}{2}(\log_q \frac{x}{\xi} - 1)\log_q \frac{x}{\xi}} \varphi(\xi) \frac{d\xi}{\xi},$$

where $\rho > 0$ is chosen small enough for $f(x)$ to exist. The formal analog of $\mathcal{B}q$ is given by

$$\widehat{\mathcal{B}q}(\sum_{n\geq 0} a_n x^n) = \sum_{n\geq 0} \frac{a_n \xi^n}{q^{n(n-1)/2}}.$$

Definition 1. A series $\sum_{n\geq 0} a_n x^n$ is a q-Borel-Laplace summable series for the direction θ if it can be applied a q-Borel and q-Laplace transform relative to the direction θ and close directions.

The Theta series is the typical example of a q-Borel-Laplace summable series.

Definition 2. – A series $\sum_{n\geq 0} a_n x^n$ is of q-Gevrey type (of level 1) if it satisfies a growth condition $|a_n| \leq K q^{n(n-1)/2} A^n$ for all n and suitable constants K and A.

– A function f is q-asymptotic of level 1 to a series $\widehat{f}(x) = \sum_{n\geq 0} a_n x^n$ for the direction θ if, for suitable constants $K_\theta > 0$ and $A_\theta > 0$, the inequality

$$(*_\theta) \qquad \left| f(x) - \sum_{m=0}^{n-1} a_m x^m \right| \leq K_\theta q^{\frac{1}{2}(n^2 + \arg_q(xe^{-i\theta}))} A_\theta^n |x|^n$$

holds for all n and small enough x on the Riemann surface of Log.

The Jacobi function f_0 is q-asymptotic to the Theta series for all directions but the directions $\theta = 0 \bmod 2\pi$.

A q-asymptotic expansion is also an asymptotic expansion in the usual Poincaré sense. Hence, if it exists, it is unique and can be called the Taylor series of the function. There exist q-flat functions. However one has the following result.

Proposition 1. *The unique function to be q-flat in two different directions is the null function.*

Definition 3. A series $\widehat{f}(x) = \sum_{n\geq 0} a_n x^n$ is said q-summable of level 1 with q-sum f for the direction θ if the condition $(*_\theta)$ holds locally uniformly with respect to θ, i.e., if there exist a neighbourhood $(\theta - \varepsilon, \theta + \varepsilon)$ of θ and constants K and A such that

$$(**_\theta) \qquad \left| f(x) - \sum_{m=0}^{n-1} a_m x^m \right| \leq K q^{\frac{1}{2}(n^2 + \arg_q(xe^{-i\tilde{\theta}}))} A^n |x|^n$$

for all n, all $\tilde{\theta} \in (\theta - \varepsilon, \theta + \varepsilon)$ and all small enough x.

It results from Proposition 1 that the q-sum of level 1 of $\widehat{f}$ if it exists for the direction θ is unique.

Theorem 1. *A series is q-summable of level 1 for the direction θ if and only if it is q-Borel-Laplace summable in the direction θ and the sums are equal.*

Definition 4. A series $\widehat{f}(x) = \sum a_n x^n$ is said q-summable of level 1 (or q-Borel-Laplace summable) if it is q-summable of level 1 for all directions but locally finitely many which are called singular directions.

The series Theta is q-summable of level 1 with singular directions $\theta = 0 \mod 2\pi$.
One can extend the previous notions to any level k by substituting x^k to x or so.

3. Summability of series solutions of q-difference equations

Using the elementary operator σ_q instead of the derivation $\frac{d}{dx}$ one can define the Newton polygon of a linear q-difference operator like it can be done for a linear differential operator. A fundamental set of formal solutions was given by Adams in [1]. It is made of finite linear combinations of terms of the form

$$\widehat{f}(x) x^{\alpha} \log^m x \, e^{\frac{\mu}{2}\log^2 x} \quad \text{where} \quad \alpha \in \mathbb{C}, m \in \mathbb{N}, \mu \in \mathbb{Q}$$

and where $\widehat{f}(x)$ is a power series (possibly in a fractional power of x). The numbers μ are the different slopes of the Newton polygon $N(\Delta)$. It was proved by Carmichael [2] that when $N(\Delta)$ has the unique slope 0 then there are no exponential terms and all the power series are convergent. The origin 0 is then either an ordinary or a regular singular point.

When there is the slope 0 and a non zero slope then the origin 0 is an irregular singular point; the number of solutions without an exponential factor is equal to the length of the zero slope. Those solutions we will call the formal series solutions even though they can contain a factor $x^{\alpha} \log^m x$.

Theorem 2. *Suppose that the Newton polygon $N(\Delta)$ of a linear q-difference operator Δ admits a unique non zero slope equal to k. Then, the formal series solutions of Δ are q-summable of level k.*

Following the same kind of idea one can also define q-accelerators like it was done by J. Écalle for differential and difference equations and introduce a notion of q-accelero-summability, also called q-multisummability for finitely many levels $\mu_1, \ldots, \mu_p$.

Theorem 3. *Suppose that the Newton polygon $N(\Delta)$ of a linear q-difference operator Δ admits the non zero slopes $\mu_1, \ldots, \mu_p$. Then, the formal series solutions of Δ are q-multisummable of levels $(\mu_1, \ldots, \mu_p)$.*

Proposition 2. *q-summable series of level k are naturally given a structure of $\mathbb{C}\{x\}$-module, not a structure of algebra.*

For example, if $\widehat{f}$ is a non convergent q-summable series of level 1 then $\widehat{f}^2$ is not q-summable of any level k; however it is q-multisummable of levels $(1, 2)$.

Bibliography

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