Equations in S_n and Combinatorial Maps

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November 3, 1997

[summary by Dominique Gouyou-Beauchamps]

This talk presents a joint work with Alain Goupil (LACIM-UQAM, Montréal).

1. Counting Maps

A *partition* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is a finite non-increasing sequence of positive integers λ_i such that $\lambda_1 \geq \dots \geq \lambda_k > 0$. The non-zero terms are called the *parts* of λ and the number k of parts is the *length* of λ, denoted $\ell(\lambda)$. We also write $\lambda = 1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}$ when α_i parts of λ are equal to i $(i = 1, \dots, n)$. When the sum $\lambda_1 + \lambda_2 + \dots + \lambda_k = n$, we call n the *weight* of λ and we write $\lambda \vdash n$ or $|\lambda| = n$. The conjugacy classes C_λ of the symmetric group S_n are indexed by partitions of n which are called the cycle types of the permutations $\sigma \in C_\lambda$.

There exit relations between pairs of permutations and maps on oriented surfaces. A *map* (S, G) on a compact oriented surface S without boundary is a graph G together with an embedding of G into S such that connected components of the complement $S \setminus G$ of the embedding of G in S, called the faces of the map, are homeomorphic to discs. Multiple edges are allowed and our maps are rooted, i.e., one edge of G is distinguished. Two maps (S, G) and (S', G') are isomorphic if there exits an orientation-preserving homeomorphism $f : S \to S'$ such that $f(G) = G'$. A map is *bicolored* if its vertices are colored in black or white so that each edge is incident to one vertex of each color. A map is *unicellular* if it has one face. The *type* of a bicolored unicellular map M with n edges is a pair of partitions (λ, μ) whose parts give respective degrees of black and white vertices of M.

Proposition 1 (see [3] for more details). *Bicolored unicellular maps of type (λ, μ) are maps on a compact orientable surface of genus g which satisfy $g = g(\lambda, \mu)$. Moreover, the number $Bi(\lambda, \mu)$ of bicolored unicellular maps of type (λ, μ) with n edges is the number of pairs (σ, τ) such that $\sigma\tau = (1, 2, \dots, n)$, which is also the coefficient $c^{(n)}_{\lambda,\mu}$ that we study below.*

2. Equations in S_n

Following [7], let $z_\lambda = \prod_i \alpha_i! i^{\alpha_i}$ for a partition $\lambda = 1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}$. Then

$$\mathrm{Card}(C_\lambda) = |C_\lambda| = \frac{n!}{z_\lambda}.$$

Example. The conjugacy class T of transpositions is $T = C_{1^{n-2}2}$ and $|T| = \binom{n}{2} = \frac{n!}{(n-2)!2}$.

In this talk, we are interested with the general problem of computing the number

$$C^{\pi}_{\lambda^1,\dots,\lambda^m} = \left| \sum (\lambda^1, \dots, \lambda^m; \pi) \right|$$

of solutions $(\alpha_1, \dots, \alpha_m) \in C_{\lambda^1} \times \cdots \times C_{\lambda^m}$ of the equation $\alpha_1\alpha_2\cdots\alpha_m = \pi$ where π is any fixed permutation of S_n and where $\langle\alpha_1, \dots, \alpha_m\rangle$ acts transitively on $\{1, 2, \dots, n\}$.

Example. Factorization of any n-cycle into transpositions

$$C^{(n)}_{T^{n-1}} = |\{(\tau_1, \dots, \tau_{n-1}) \text{ transpositions such that } \tau_1\cdots\tau_{n-1} = (1, 2, \dots, n)\}| = n^{n-2}.$$

If $\alpha \in C_\lambda$ and $\alpha = \tau_1\cdots\tau_k$, then we have $k \geq n - \ell(\lambda) = \sum_{i=1}^{\ell(\lambda)}(\lambda_i - 1)$ and parity of α is given by parity of $n - \ell(\lambda)$. Thus if $\alpha_1\alpha_2\cdots\alpha_m = \pi$, we have the first necessary condition for existence of solutions in $\sum(\lambda^1, \dots, \lambda^m; \pi)$:

$$\sum_{i=1}^{m} \left(n - \ell(\lambda^i)\right) \equiv n - \ell(\pi) \bmod 2.$$

If $\langle\alpha_1, \dots, \alpha_m\rangle$ acts transitively on $\{1, 2, \dots, n\}$, the underlying graph is connected.

Example. $\tau_1\cdots\tau_m = 1$. We need $m = 2n - 2$ transpositions: $n - 1$ transpositions to get an n-cycle and one connected component, and $n - 1$ transpositions to return to 1.

Proposition 2. *Let $(\alpha_1, \dots, \alpha_m)$ be in $C_{\lambda^1} \times \cdots \times C_{\lambda^m}$. If $\langle\alpha_1, \dots, \alpha_m\rangle$ acts transitively and if $\alpha_1\cdots\alpha_m = 1$, then $\sum_{i=1}^{m}\left(n - \ell(\lambda^i)\right) \geq 2n - 2$.*

Definition 1. The genus of m partitions $(\lambda^1, \dots, \lambda^m)$ of weight n is the non negative integer g defined by the equation

$$\sum_{i=1}^{m} \left(n - \ell(\lambda^i)\right) = 2n - 2 + 2g.$$

For non transitive systems, we want to compute the number

$$d^{\pi}_{\lambda^1,\dots,\lambda^m} = \left|\widehat{\sum}(\lambda^1, \dots, \lambda^m; \pi)\right|$$

of solutions $(\alpha_1, \dots, \alpha_m) \in C_{\lambda^1} \times \cdots \times C_{\lambda^m}$ of the equation $\alpha_1\alpha_2\cdots\alpha_m = \pi$ where π is any fixed permutation of S_n.

We remark that $\widehat{\sum}(\lambda^1, \dots, \lambda^m; \pi) = \text{Set}\left(\sum(\lambda^1, \dots, \lambda^m; \pi)\right)$. From this observation we deduce the exponential generating function (in the variables $(p^{(i)}_j)$, $j \geq 1$, $1 \leq i \leq m$) of $\widehat{\sum}(\lambda^1, \dots, \lambda^m; \pi)$ for a fixed m:

$$\sum_{n\geq 0} \frac{z^n}{n!} \sum_{\lambda_1,\dots,\lambda_n,\pi} d^{\pi}_{\lambda^1,\dots,\lambda^m} P^{(1)}_{\lambda_1} P^{(2)}_{\lambda_2} \cdots P^{(m)}_{\lambda_m} = \exp\left(\sum_{n\geq 1} \frac{z^n}{n!} \sum_{\lambda_1,\dots,\lambda_n,\pi} c^{\pi}_{\lambda^1,\dots,\lambda^m} P^{(1)}_{\lambda_1} P^{(2)}_{\lambda_2} \cdots P^{(m)}_{\lambda_m}\right)$$

where $P^{(i)}_\lambda = p^{(i)}_{\lambda_1}\cdots p^{(i)}_{\lambda_k}$. Hence, in order to obtain the generating function, we only have to know all the $d^{\pi}_{\lambda^1,\dots,\lambda^m}$.

3. **Theory of Characters**

Detailed proofs for this section can be found in [5, 6].

Theorem 1 (Frobenius formula). *Let G be a finite group. The number of solutions $(g_1, \dots, g_m) \in C_{\lambda^1} \times \cdots \times C_{\lambda^m}$ of the equation $g_1\cdots g_m = 1$ is*

$$\frac{|C_1|\cdots|C_m|}{|G|} \sum_{\chi} \frac{\chi(C_1)\cdots\chi(C_m)}{[\chi(1)]^{m-2}}$$

where the sum is extended over the irreducible characters of G.

If G is the symmetric group S_n, the irreducible characters are $\{\chi^\mu\}_{\mu\vdash n}$ and $\chi^\mu(C_\lambda)$ can be computed by the Murnaghan-Nakayama rule

$$\chi^\mu(C_\lambda) = \chi^\mu_\lambda = \sum_{T\in\mathrm{T}(\lambda,\mu)} \prod_{S\in T} (-1)^{h(S)}.$$

Hence theoretically $d^\pi_{\lambda^1,\ldots,\lambda^m}$ can be computed since it can be rewritten as

$$d^\pi_{\lambda^1,\ldots,\lambda^m} = \frac{|C_{\lambda_1}|\cdots|C_{\lambda_m}|}{n!} \sum_{\mu\vdash n} \frac{\chi^\mu_{\lambda_1}\cdots\chi^\mu_{\lambda_m}\chi^\mu_\pi}{[f^\mu]^{m-1}}$$

where f^μ is the number of standard Young tableaux of shape μ. But it is hopeless for $n \geq 15$.

4. Results for Genus 0

The following results are known.

Theorem 2 (Dénès theorem). *Factorization of an n-cycle into $n-1$ tranpositions:* $C^{(n)}_{T^{n-1}} = n^{n-2}$.

Theorem 3 (Hurwitz formula). *Factorization of α of cycle-type $(\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})$ into a minimal product of $n + \ell(\alpha) - 2$ transpositions acting transitively on $\{1, 2, \ldots, n\}$:*

$$C^\alpha_{T^{n+\ell(\alpha)-2}} = n^{\ell(\alpha)-3}(n + \ell(\alpha) - 2)! \prod_{i=1}^{k} \frac{\alpha_i^{\alpha_i}}{(\alpha_i - 1)!}.$$

A new bijective proof without using theory of characters is given in [2].

Theorem 4 (Tree cacti of Goulden and Jackson [4]).

$$C^{(n)}_{\lambda_1,\ldots,\lambda_m} = n^{m-1} \prod_{i=1}^{m} \frac{1}{\ell(\lambda_i)} \binom{\ell(\lambda_i)}{\alpha^i_1, \ldots, \alpha^i_m}$$

$$\textit{with } \lambda^i = 1^{\alpha^i_1} 2^{\alpha^i_2} \cdots n^{\alpha^i_n} \textit{ and minimality: } \sum_{i=1}^{m} (n - \ell(\lambda^i)) = n - 1.$$

The proof uses a recursive decomposition of tree cacti and the Lagrange-Good inversion.

5. Our Main Theorem

Theorem 5 (A. Goupil and G. Schaeffer). *Factorization of an n-cycle into two permutations of cycle-types λ and μ with $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\mu = (\mu_1, \ldots, \mu_k)$ and $\ell(\lambda) + \ell(\mu) = n + 1 - 2g$:*

$$C^{(n)}_{\lambda,\mu} = \frac{n}{z_\lambda z_\mu 2^{2g}} \sum_{g_1+g_2=g} (\ell(\lambda) - 1 + 2g_1)!(\ell(\mu) - 1 + 2g_2)! S_{g_1}(\lambda) S_{g_2}(\mu)$$

$$\textit{with } S_g(\lambda) = \sum_{i_1+\cdots+i_{\ell(\lambda)}=g} \prod_{k=1}^{\ell(\lambda)} \binom{\lambda_k}{2i_k + 1}.$$

$$\text{If } g = 0, \qquad C^{(n)}_{\lambda,\mu} = \frac{n(\ell(\lambda) - 1)!(\ell(\mu) - 1)!}{z_\lambda z_\mu} \prod_{i=1}^{k} \lambda_i \prod_{j=1}^{k} \mu_j.$$

In [1] this simple expression was derived. This coefficient was later interpreted combinatorially by Goulden and Jackson [4] as the number of unicellular rooted bicolored maps with n edges on a surface of genus zero, the vertices of each color having degree distribution given by λ and μ respectively, that is the case where $m = 2$ in theorem 4.

$$\text{If } g = 1, \qquad C^{(n)}_{\lambda,\mu} = \frac{n(\ell(\lambda)-1)!(\ell(\mu)-1)!}{2z_\lambda z_\mu}\left[\binom{\ell(\lambda)+1}{2}\sum_{i=1}^{k}\binom{\lambda_i}{3} + \binom{\ell(\mu)+1}{2}\sum_{i=1}^{k}\binom{\mu_i}{3}\right].$$

Survey of the proof of Theorem 5:

1. Using explicit expressions for characters of the symmetric group, we give the following formula

$$c^{(n)}_{\lambda,\mu} = \frac{n}{z_\lambda z_\mu}\sum_{r=0}^{n-1}(-1)^r r!(n-1-r)!\chi_\lambda^{1^r(n-r)}\chi_\mu^{1^r(n-r)}. \tag{1}$$

2. The evaluation of some characters are given as weighted summations over set of "quasi-painted diagrams".
3. We use a bijection to replace quasi-painted diagrams by properly "painted diagrams" and we rewrite Formula (1) as a weighted summation over some "painted diagram matchings".
4. The introduction of "connected components" of diagram matchings allows to set apart the diagram matching from its painting and to show that the weight depends only on the painting. This is used to apply a sign reversing involution.
5. As expected, the fix-points yield positive contributions. These contributions count "colorings" of the diagram matchings.
6. We show that colored diagrams are enumerated by formula of Theorem 5.

6. Corollary for Genus > 0

$$C^{(n)}_{T^{n-1+2g}} = \frac{n^{n-2+2g}}{(n-1)!2^{2g}}\sum_{c_1,\dots,c_{n-1}}\binom{n-1+2g}{c_1,\dots,c_{n-1}} \sim_{n\to\infty} \frac{n^{n-2+5g}}{g!24^g}$$

where the sum is taken over the odd c_i such that $\sum c_i = n-1+2g$.

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