Multivariate Lagrange Inversion

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May 25, 1998

[summary by Danièle Gardy]

Abstract

A new formulation of Lagrange inversion for several variables will be described which does not involve a determinant. This formulation is convenient for the asymptotic investigation of numbers defined by Lagrange inversion. Examples of tree problems where the number of vertices of degree k are counted and where vertices are 2-colored will be given. Non-crossing partitions give another example and the Meir-Moon formula for powers of an inversion is a special case.

1. Running Example

Consider a rooted plane tree where internal vertices can have two or three sons and are green or red, according to the following rules: (an example of such a tree is given below.)

- a green vertex has three children; one is red and the other two are green;
- a red vertex has two children, one of each color, and the left one is red.

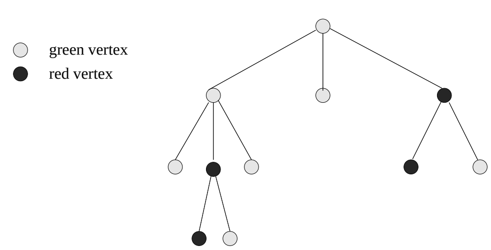
Enumeration of such trees is best done by taking into account the colors of the vertices: let x_1 and x_2 mark the green and red vertices, and define $w_1(x_1, x_2)$ and $w_2(x_1, x_2)$ as the functions enumerating the trees whose root is green (resp. red). These functions satisfy the system of equations

$$w_1(x_1, x_2) = x_1(1 + 3w_1^2 w_2); \qquad w_2(x_1, x_2) = x_2(1 + w_1 w_2).$$

Introducing the vectors $\underline{x} = (x_1, x_2)$ and $\underline{w} = (w_1, w_2)$ and the functions $f_1(\underline{w}) = 1 + 3w_1^2 w_2$ and $f_2(\underline{w}) = 1 + w_1 w_2$, one obtains the system $w_1(\underline{x}) = x_1 f_1(\underline{w}); w_2(\underline{x}) = x_2 f_2(\underline{w})$. Such equations are very similar to those that can be solved in one dimension by Lagrange inversion, and it is natural to try and solve them with a suitable extension.

2. Multivariate Lagrange Inversion

In one dimension, Lagrange inversion is used for implicit equations of the type $w(x) = xf(w(x))$, with $f(0) \neq 0$: It relates the coefficients of a solution $w(x)$, or of a function of $w(x)$, as formal



power series, to the coefficients of the simpler function f:

$$[x^n]w(x) = \frac{1}{n}[t^{n-1}]f(t)^n; \qquad [x^n]g(w(x)) = \frac{1}{n}[t^{n-1}]g'(t)f^n(t).$$

Extensions to the multivariate case have been considered for some time; surveys can be found in the paper written some twelve years back by Gessel [6], or in the recent book by Bergeron, Labelle and Leroux [4]. The version presented below is due to Good [7]:

Theorem 1. *Let $\underline{x}$ be a d-dimensional vector, $g(\underline{x})$ and $f_i(\underline{x})$ $(1 \leq i \leq d)$ be formal power series in $\underline{x}$, s.t. $f_i(\underline{0}) \neq 0$. Then the equations $w_i = x_i f_i(\underline{w})$ uniquely determine the w_i as formal power series in $\underline{x}$, and*

$$[\underline{t}^{\underline{n}}]g(\underline{w}(\underline{t})) = [\underline{x}^{\underline{n}}]\left(g(\underline{x})\underline{f}^{\underline{n}}(\underline{x}) \left\|\delta_{i,j} - \frac{x_i \partial f_j(\underline{x})}{f_j(\underline{x})\partial x_i}\right\|\right),$$

with $\delta_{i,j}$ the Kronecker symbol, $||A||$ the determinant of the matrix A, $\underline{f} = (f_1, \ldots, f_d)$, and $\underline{f}^{\underline{n}} = f_1^{n_1} \cdots f_d^{n_d}$.

The determinant in this formula leads to trouble when one tries to get asymptotic information from it. Let us consider the univariate case to see what the problem is.

For $d = 1$, Good's formula applied to the equation $w(x) = xf(w(x))$ gives an identity equivalent to the one presented above:

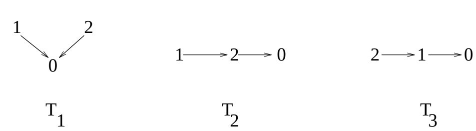
$$[x^n]w(x) = [t^{n-1}]\left(f^n(t)\left(1 - t\frac{f'(t)}{f(t)}\right)\right). \tag{1}$$

When one wishes to obtain asymptotics, a natural tool is the saddle-point method, well suited to approximating coefficients of (variations on) large powers of functions; see for example [5] for a summary of results in this area. The idea is to use Cauchy's formula $[z^n]F(z) = \oint F(z)z^{-n-1}dz$, for $F(z) = f(z)^n(1 - zf'(z)/f(z))$, with an integration path that is a circle going through the saddle-point ρ_0; ρ_0 is itself is a perturbation of the saddle-point ρ_1 that appears in the evaluation of the simpler coefficient $[x^n]f^n(x)$. Now ρ_1 is defined as the solution of the equation $1 - xf'(x)/f(x) = 0$, i.e. the integrand of the right part of (1) becomes zero close to ρ_0!

With care, it should be possible to work this out for one variable, but the outlook for a multi-dimensional extension is not favorable, as we can expect cancellation of the determinant close to the integration paths. Instead, Bender and Richmond have proposed a new multivariate version, better suited to asymptotics; this formula will use the derivatives of a vector wrt a directed graph.

3. **Differentiating a Vector wrt a Directed Graph**

To define the partial of a vector relative to a directed graph, consider all trees with vertices $0, 1, \ldots, d$ and edges directed to 0. There are $(d+1)^{d-1}$ such trees; for example for $d = 2$ there are three trees:



Now the derivative of a $(d+1)$-dimensional function $\underline{f}$ according to such a tree is a product on $(d+1)$ terms, where f_i is differentiated according to the incoming edges into the vertex labelled by i; this is best explained on the above example, with $\underline{f} = (f_0, f_1, f_2)$:[1]

[1] Although the definition is more general, trees are the only graphs considered here.

$$\frac{\partial f}{\partial T_1} = \frac{\partial^2 f_0}{\partial x_1 \partial x_2} \cdot f_1 \cdot f_2; \qquad \frac{\partial f}{\partial T_2} = \frac{\partial f_0}{\partial x_2} \cdot f_1 \cdot \frac{\partial f_2}{\partial x_1}; \qquad \frac{\partial f}{\partial T_3} = \frac{\partial f_0}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdot f_2.$$

4. The New Inversion Formula

Theorem 2. *Under the assumptions of the former theorem,*

$$[\underline{t}^{\underline{n}}]g(\underline{w}(\underline{t})) = \left(\prod_{i=1}^{d} n_i\right)^{-1} [\underline{x}^{\underline{n}-\underline{1}}] \sum_T \frac{\partial(g, f_1^{n_1}, \ldots, f_d^{n_d})}{\partial T},$$

where the sum is on the set of trees with $d+1$ vertices.

Proof. This result is proven in [3]; it relies on the simple formula $n[x^{n-1}]f = [x^n]\partial f/\partial x$ and on the expansion of a determinant. The terms are all positive as soon as the functions f_i and g have positive coefficients; hence the coefficient $[\underline{t}^{\underline{n}}]g(\underline{w}(\underline{t}))$, as a sum of $(d+1)^{d-1}$ such terms, is itself positive and there are no more cancellations. ☐

What do we obtain for the first values of d? For $d = 1$, the only tree is $1 \to 0$ and one gets back the classical formula. For $d = 2$, $g(t_1, t_2)$ is a function of two variables and

$$\begin{aligned}
[x_1^{n_1} x_2^{n_2}]g(w_1(x_1,x_2), w_2(x_1,x_2)) &= \frac{1}{n_1 n_2}[t_1^{n_1-1} t_2^{n_2-1}] \sum_{T \in \{T_0, T_1, T_2\}} \frac{\partial(g, f_1^{n_1}, f_2^{n_2})}{\partial T} \\
&= \frac{1}{n_1 n_2}[t_1^{n_1-1} t_2^{n_2-1}] \left(\frac{\partial^2 g}{\partial t_1 \partial t_2} f_1^{n_1} f_2^{n_2} + \frac{\partial g}{\partial t_2} f_1^{n_1} \frac{\partial (f_2^{n_2})}{\partial t_1} + \frac{\partial g}{\partial t_1} \frac{\partial (f_1^{n_1})}{\partial t_2} f_2^{n_2} \right) \\
&= \frac{1}{(n_1 - 1)(n_2 - 1)}[t_1^{n_1} t_2^{n_2}] \left(f_1^{n_1} f_2^{n_2} h \right),
\end{aligned}$$

with (f_1 and f_2 are strictly positive at the saddle-points)

$$h := \frac{\partial^2 g}{\partial t_1 \partial t_2} + n_2 \frac{\partial g}{\partial t_2} \frac{\partial f_2}{\partial t_1} \frac{1}{f_2} + n_1 \frac{\partial g}{\partial t_1} \frac{\partial f_1}{\partial t_2} \frac{1}{f_1}.$$

For general d, there is no determinant here, but a finite (although large!) sum of terms, each of which can be evaluated individually. The asymptotic value of $[\underline{t}^{\underline{n}}]g(w(\underline{x}))$ is obtained by adding the individual asymptotic values of the $(d+1)^{d-1}$ terms.

It is possible to obtain a univariate local limit theorem for the number of red vertices in trees having a fixed number of vertices, or a bivariate local limit theorem for the joint distribution of the numbers of red and green vertices.

5. Local Limit Theorem

The usual approach towards a limiting theorem is through the covariance matrix (see for example a former paper by the same authors [1]); checking the non-degeneracy of this matrix leads to intricate conditions, which the authors try to bypass, by requiring instead the existence of a multivariate saddle-point. A local limit theorem holds whenever the functions $g(\underline{x})$ and $f_i(\underline{x})$ $(1 \le i \le d)$ are analytic; there is also an existence condition on the exponents of the variables in the functions whose coefficients we are studying. Formally, this involves the lattice generated by the exponents $\underline{k}$ for which the coefficient of $\underline{t}^{\underline{k}}$ in f_i is not zero; see [2] for a precise formulation.

For example, for the colored trees presented in Section 1, the only non-zero coefficients are obtained, besides $\underline{k} = (0,0)$, for $\underline{k} = (2,1)$ in f_1, and for $\underline{k} = (1,1)$ in f_2. The lattice generated by $\{(1,1),(2,1)\}$ is $\mathbb{N}^2$; hence all the terms $t_1^{i_1} t_2^{i_2}$ will appear in the function $f_1^{k_1} f_2^{k_2}$.

The saddle-point condition is that we should be able to solve the system of d equations $\{k_i = \sum_{1<l<d} k_l \partial \log f_l / \partial \log \gamma_i\}$ (with $\gamma_i = e^{s_i}$).

We give the equations below for two variables, the better to understand what is going on, but it should be understood that it is more general and applies to d dimensions.

At some point, we have to compute a coefficient $[t_1^{n_1} t_2^{n_2}](h f_1^{n_1} f_2^{n_2})$, where the functions h, f_1 and f_2 are on the variables t_1 and t_2. The way to do this is through a saddle-point approximation; more specifically we shall look at $[t_1^{k_1} t_2^{k_2}](h f_1^{n_1} f_2^{n_2})$ for k_1 and k_2 of the same order as n_1 and n_2, but not necessarily equal. This coefficient can be written, by Cauchy's formula, as $\frac{1}{(2i\pi)^2} \oint \oint e^{h(t_1,t_2)} dt_1 dt_2$, with $h = n_1 \log f_1 + n_2 \log f_2 - k_1 \log t_1 - k_2 \log t_2$. Now the saddle-points are defined by the two equations $\partial h / \partial t_1 = 0$ and $\partial h / \partial t_2 = 0$, which give the two-dimensional system

$$k_1 = n_1 t_1 \frac{\partial f_1}{\partial t_1} \frac{1}{f_1} + n_2 t_1 \frac{\partial f_2}{\partial t_1} \frac{1}{f_2}; \qquad k_2 = n_1 t_2 \frac{\partial f_1}{\partial t_2} \frac{1}{f_1} + n_2 t_2 \frac{\partial f_2}{\partial t_2} \frac{1}{f_2}.$$

Applied to our running example, this gives the system in t_1 and t_2

$$k_1 = n_1 \frac{6 t_1^2 t_2}{1 + 3 t_1^2 t_2} + n_2 \frac{t_1 t_2}{1 + t_1 t_2}; \qquad k_2 = n_1 \frac{3 t_1^2 t_2}{1 + 3 t_1^2 t_2} + n_2 \frac{t_1 t_2}{1 + t_1 t_2}.$$

Define $\rho := k_1 / k_2$; $\rho \in]1, 2[$. Solving, we get

$$t_1 = \frac{(\rho - 1)^2}{3(2 - \rho)} =: r_1; \qquad t_2 = \frac{3(2 - \rho)^2}{(\rho - 1)^3} =: r_2.$$

This gives $(k_1, k_2) = n(\rho/(1+\rho), 1/(1+\rho))$. The covariance matrix is obtained by differentiation of $\log f$, where $f := f_1^{n_1} f_2^{n_2}$, with f_1 and f_2 defined in Section 1. For example $B_{1,1}$ is the value of $t_1 \partial(\log f)/\partial t_1 + t_1^2 \partial^2 (\log f)/\partial t_1^2$, taken at the point (r_1, r_2), which gives $B_{1,1} = n(\rho - 1)(4 + 2\rho - \rho^2)/\rho(1+\rho)$. Similar computations give the other components of the covariance matrix:

$$n \frac{\rho - 1}{\rho(1+\rho)} \begin{bmatrix} 4 + 2\rho - \rho^2 & 2 + 2\rho - \rho^2 \\ 2 + 2\rho - \rho^2 & 1 + 2\rho - \rho^2 \end{bmatrix}.$$

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