Monodromy of Polylogarithms

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[summary by David M. Bradley]

Abstract

Generalized polylogarithms are complex, multivalued functions with singularities at $z = 0$ and $z = 1$. We calculate the monodromy at the two singularities. As opposed to the classical polylogs [11, 12], the monodromy of generalized polylogs involves the so-called "multiple zeta values," [14] which play an important role in number theory, knot theory [4, 6, 5, 10], and physics [7, 9]. Via monodromy of polylogs, Radford [13] showed that the C-algebra of polylogs is isomorphic to the C-algebra of non-commutative polynomials in two variables—a "shuffle algebra" freely generated by the so-called Lyndon words. Here, monodromy is used to give an induction proof of the linear independence of the polylogarithms. We also obtain a Gröbner basis of the polynomial relations between "multiple zeta values" using the techniques of non-commutative algebra. By expressing multiple zeta values in terms of the Gröbner basis, one obtains symbolic algebraic proofs of relations between multiple zeta values.

1. Polylogarithms and Combinatorics on Words

Let $X = \{x_0, x_1\}$. To any word $w = x_0^{s_1-1}x_1x_0^{s_2-1}x_1\cdots x_0^{s_k-1}x_1$ we associate the multi-index $s = (s_1, s_2, \ldots, s_k)$ and define the generalized polylogarithm

$$\mathrm{Li}_w(z) = \mathrm{Li}_s(z) = \sum_{n_1>n_2>\cdots>n_k>0} \frac{z^{n_1}}{n_1^{s_1}n_2^{s_2}\cdots n_k^{s_k}}.$$

The associated multiple zeta value is $\zeta_w = \zeta(s) = \mathrm{Li}_w(1) = \mathrm{Li}_s(1)$. The *shuffle product* is defined on words by the recursion

$$xu ш yv = x(u ш yv) + y(xu ш v),$$

where $x, y \in X$ and u and v are words on X. We can extend the shuffle product linearly to the non-commutative polynomials $\mathbb{Q}\langle X\rangle$. The resulting polynomial algebra, denoted $\mathrm{Sh}_{\mathbb{Q}}(X)$ is commutative and associative.

The Lyndon words L are those non-empty words on X that are inferior to each of their right factors in the lexicographical order. They are algebraically independent and generate $\mathrm{Sh}_{\mathbb{Q}}(X)$, thus forming a transcendence basis. More precisely, a theorem of Radford [13] states that the algebra $\mathrm{Sh}_{\mathbb{Q}}(X)$ is isomorphic to the polynomial algebra generated by the Lyndon words, i.e. $\mathbb{Q}[L]$.

2. Relations between Multiple Zeta Values

There are countless relations between multiple zeta values [1, 3, 2]. We content ourselves here with providing only two examples:

$$\zeta(2,1)=\zeta(3) \quad \text{and} \quad \zeta(2,2,1)=-\frac{11}{2}\zeta(5)+3\zeta(2)\zeta(3).$$

It turns out that a large class of relations can be explained by the collision of two distinct shuffles obeyed by the multiple zeta values. We've already seen one type of shuffle. It provides relations of the form $\zeta_{u ш v}=\zeta_u\zeta_v$. A second type of shuffle provides relations of the form $\zeta_{u*v}=\zeta_u\zeta_v$ and is defined by the recursion

$$(s_1,s)*(t_1,t)=(s_1,s*(t_1,t))+(t_1,(s_1,s)*t)+(s_1+t_1,s*t),$$

where we have used the multi-index notation $s=(s_2,s_3,\ldots,s_k)$, $t=(t_2,t_3,\ldots,t_r)$ of Section 1. With a slight abuse of notation, we define a map $\zeta: w\to\zeta_w$, extended linearly in the natural way to $\mathbb{Q}\langle X\rangle$. Then ζ is a $\mathbb{Q}$-algebra homomorphism which respects both shuffle products. Thus, if I is the ideal generated by the words $u ш v-u*v$, then $I\subseteq \ker\zeta$. We can compute a Gröbner basis for the ideal I up to any given order using only symbolic computation. The first relation above is the unique basis element of order 3. The second relation above is one of five basis elements of order 5.

3. Monodromy of Polylogarithms

To compute the monodromy, we use the standard keyhole contours about the two singularities $z=0$ and $z=1$. The monodromy is given by

$$M_0\mathrm{Li}_{wx_0}=\mathrm{Li}_{wx_0}+2\pi i\mathrm{Li}_w+\cdots$$
$$M_1\mathrm{Li}_{wx_1}=\mathrm{Li}_{wx_1}-2\pi i\mathrm{Li}_w+\cdots,$$

where the remaining terms are linear combinations of polylogarithms coded by words of lengths less than the length of w. For example, using the computational package Axiom, we find that

$$M_1\mathrm{Li}_{x_0}=\mathrm{Li}_{x_0},\quad M_1\mathrm{Li}_{x_1}=\mathrm{Li}_{x_1}-2\pi i,\quad M_1\mathrm{Li}_{x_0x_1}=\mathrm{Li}_{x_0x_1}-2\pi i\mathrm{Li}_{x_0},$$

and so on. The generating series of the generalized polylogarithms is

$$L(z)=\sum_{w\in X^*} w\mathrm{Li}_w(z),$$

with the convention that $\mathrm{Li}_{x_0^n}(z)=(\log z)^n/n!$. Drinfel'd's differential equation [8, 9]

$$\frac{d}{dz}L(z)=\left(\frac{x_0}{z}+\frac{x_1}{1-z}\right)L(z),$$

is satisfied, with boundary condition $L(\epsilon)=\exp(x_0\log\epsilon)+O(\sqrt{\epsilon})$ as $\epsilon\to 0+$. It turns out that L is a Lie exponential, and this fact can be used to obtain asymptotic expansions of the generalized polylogarithms at $z=1$.

4. Independence of Polylogarithms

Theorem 1. *The functions* Li_w *with* $w\in X^*$ *are* $\mathbb{C}$*-linearly independent.*

Corollary 1. *The* $\mathbb{C}$*-algebra generated by the* Li_w *is isomorphic to* $\mathrm{Sh}_{\mathbb{C}}(X)$. *By Radford's theorem, the generalized polylogarithms coded by Lyndon words form an infinite transcendence basis.*

Corollary 2. *Each generalized polylogarithm* Li_w *has a unique representation as a* $\mathbb{Q}$*-polynomial in polylogarithms coded by Lyndon words. The classical* [11, 12] *polylogarithms* Li_k*, which are coded by the Lyndon words* $x_0^{k-1}x_1$*, are algebraically independent.*

Proof of Theorem 1. Given $n \geq 0$, assume that

$$\sum_{|w|\leq n} \lambda_w \mathrm{Li}_w = 0, \qquad \lambda_w \in \mathbb{C}, \tag{1}$$

where $|w|$ denotes the length of the word w. We prove by induction on n that $\lambda_w = 0$ for all w, the case $n = 0$ being trivial. Rewrite (1) as

$$\lambda_1 + \sum_{|u|<n} \lambda_{ux_0} \mathrm{Li}_{ux_0} + \sum_{|u|<n} \lambda_{ux_1} \mathrm{Li}_{ux_1} = 0.$$

Applying the operators $(M_0 - \mathrm{Id})$ and $(\mathrm{Id} - M_1)$ on this latter expression, yields two new linear relations

$$\begin{cases} 2\pi i \sum_{|u|=n-1} \lambda_{ux_0} \mathrm{Li}_u + \sum_{|u|<n-1} \mu_u \mathrm{Li}_u & = 0, \\ 2\pi i \sum_{|u|=n-1} \lambda_{ux_1} \mathrm{Li}_u + \sum_{|u|<n-1} \nu_u \mathrm{Li}_u & = 0, \end{cases}$$

for certain coefficients μ_u and ν_u. By the induction hypothesis, the coefficients λ_{ux_0} and λ_{ux_1} with $|u| = n - 1$ all vanish (as well as the coefficients μ_u and ν_u). Consequently,

$$\sum_{|w|\leq n-1} \lambda_w \mathrm{Li}_w = 0,$$

whence $\lambda_w = 0$ for all w, again by the induction hypothesis. □

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