Multidimensional Polylogarithms

David M. Bradley
Dalhousie University, Canada

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[summary by Hoang Ngoc Minh]

1. Introduction

Recently, several extensions of polylogarithms, Euler sums (or multiple harmonic sums) and Riemann zeta functions have been introduced. These have arisen in number theory, knot theory, high-energy physics, analysis of quadtrees, control theory, ... In this talk, the author presents the multidimensional polylogarithms and their special values [1, 2]. After definitions related to multidimensional polylogarithms (Section 2), results, conjectures and combinatorial aspects concerning unit Euler sums and unsigned Euler sums are discussed (Section 3). Integral representations are also pointed out to understand multidimensional polylogarithms (Section 4).

2. Definitions

Definition 1. The *multidimensional polylogarithms* (MDPs) are defined as follows

$$\lambda\binom{s_1,\ldots,s_k}{b_1,\ldots,b_k} = \prod_{j=1}^{k}\sum_{\nu_j\geq 1}\frac{b_j^{-\nu_j}}{(\nu_j+\cdots+\nu_k)^{s_j}}.$$

k is the *depth* and $s = s_1 + \cdots + s_k$ is the *weight* of $\lambda\binom{s_1,\ldots,s_k}{b_1,\ldots,b_k}$.

- When $k = 0$, by convention $\lambda(\{\}) = 1$;
- When $k = 1$, s is a positive integer and $|b| \geq 1$, one get the usual *polylogarithm*

$$\lambda\binom{s}{b} = \sum_{\nu\geq 1}\frac{b^{-\nu}}{\nu^s} = \mathrm{Li}_s(1/b).$$

  The classical *Riemann zeta function* is obtained in the special case where $b = 1$.
- When $k > 1$, let $n_j = \sum_{i=j}^{k}\nu_i$ and $b_j = \prod_{i=1}^{j} a_i$. Then

$$\lambda\binom{s_1,\ldots,s_k}{b_1,\ldots,b_k} = \sum_{n_1>\cdots>n_k>0}\frac{a_1^{-n_1}\cdots a_k^{-n_k}}{n_1^{s_1}\cdots n_k^{s_k}}.$$

  * If each $a_j = 1$ then these sums are called *Euler sums*;
  * If each $a_j = \pm 1$ then they are called *alternating Euler sums*.

Definition 2. The *unit Euler sum* is defined as follows

$$\mu(b_1,\ldots,b_k) = \lambda\binom{1,\ldots,1}{b_1,\ldots,b_k} = \prod_{j=1}^{k}\sum_{\nu_j\geq 1}\frac{b_j^{-\nu_j}}{(\nu_j+\cdots+\nu_k)}.$$

Definition 3. The *unsigned Euler sum* is defined as follows

$$\lambda_b(s_1,\ldots,s_k) = \lambda\begin{pmatrix} s_1,\ldots,s_k \\ b,\ldots,b \end{pmatrix} = \prod_{j=1}^{k}\sum_{\nu_j\geq 1}\frac{b^{-\nu_j}}{(\nu_j+\cdots+\nu_k)^{s_j}}.$$

– When $b=1$, λ_1 is often called the unsigned Euler sum or *multiple zeta value* (MZV)

$$\lambda_1(s_1,\ldots,s_k) = \zeta(s_1,\ldots,s_k) = \sum_{n_1>\cdots>n_k>0}\frac{1}{n_1^{s_1}\cdots n_k^{s_k}}.$$

– When $b=2$, λ_2 represents an iterated sum extension of the polylogarithm with argument $1/2$, and plays a crucial role in computing the MZVs.

3. Special Values of MDPs

Theorem 1. *Let p and q satisfy $1/p+1/q=1$. If in addition, $p>1$, or $p\leq -1$, then for any nonnegative integer k,*

$$\mu(\{p\}^k) = \frac{(\log q)^k}{k!}.$$

The proof is done by coefficient extraction in the generating function $\sum_{k\geq 0} x^k\mu(\{p\}^k)$.

Theorem 2. *Let $A_r = \mathrm{Li}_r(1/2), P_r = (\log 2)^r/r!, Z_r = (-1)^r\zeta(r)$. Then, for $m\geq 1, n\geq 0$*

$$\mu(\{-1\}^m,1,\{-1\}^n) = (-1)^{m+1}\sum_{k=0}^{m}\binom{n+k}{k}A_{k+n+1}P_{m-k} + (-1)^{n+1}\sum_{k=0}^{n}\binom{m+k}{m}Z_{k+m+1}P_{n-k}.$$

The proof of this theorem can be done via the *duality principle* (see Section 4).

For any nonnegative integer k, the following identities provide nested sum extensions of Euler's $\zeta(2),\zeta(4),\zeta(6)$ and $\zeta(8)$ evaluations, respectively

$$\zeta(\{2\}^k) = \frac{2(2\pi)^{2k}}{(2k+1)!}\left(\frac{1}{2}\right)^{2k+1},$$

$$\zeta(\{4\}^k) = \frac{4(2\pi)^{4k}}{(4k+2)!}\left(\frac{1}{2}\right)^{2k+1},$$

$$\zeta(\{6\}^k) = \frac{6(2\pi)^{6k}}{(6k+3)!},$$

$$\zeta(\{8\}^k) = \frac{8(2\pi)^{8k}}{(8k+4)!}\left[\left(1+\frac{1}{\sqrt{2}}\right)^{4k+2} + \left(1-\frac{1}{\sqrt{2}}\right)^{4k+2}\right].$$

In general, for any positive integer n, $\varepsilon = \mathrm{e}^{i\pi/n}$, one has

$$\sum_{k\geq 0}(-1)^k x^{2kn}\zeta(\{2n\}^k) = \prod_{j=0}^{n-1}\frac{\sin(\pi x\varepsilon^j)}{\pi x\varepsilon^j}.$$

Theorem 3 (Zagier's conjecture [6]).

$$\zeta(\{3,1\}^n) = 4^{-n}\zeta(\{4\}^n) = \frac{2\pi^{4n}}{(4n+2)!}.$$

Conjecture 1.

$$\zeta(2,\{3,1\}^n) = 4^{-n}\sum_{k=0}^{n}(-1)^k\zeta(\{4\}^{n-k})\left[(4k+1)\zeta(4n+2) - 4\sum_{j=1}^{k}\zeta(4j-1)\zeta(4k-4j+3)\right].$$

In practice, one would like to know which unsigned Euler sums can be expressed in terms of lower depth sums. When the sum can be expressed, it is said to "reduce". Hoang Ngoc Minh and Michel Petitot have implemented in `AXIOM` an algorithm to reduce the MZVs via a table of Gröbner basis of these sums at fixed weight [5]. Here, the authors also get the following

Theorem 4. *For any positive integer k,*

$$\zeta(s_1,\ldots,s_k) + (-1)^k\zeta(s_k,\ldots,s_1)$$

reduces to lower depth MZVs.

The following theorem gives Crandall's recurrence for unsigned Euler sums $\zeta(\{s\}^k)$ and it can be proved by coefficient extraction in the generating function $\sum_{k\geq 0} kx^k\zeta(\{s\}^k)$.

Theorem 5 (Crandall's recurrence). *For any nonnegative integer k and $\Re(s) > 0$,*

$$k\zeta(\{s\}^k) = \sum_{j=1}^{k}(-1)^{j+1}\zeta(js)\zeta(\{s\}^{k-j}).$$

For example

$$\begin{aligned}\zeta(\{s\}) &= \zeta(s),\\ \zeta(\{s,s\}) &= \frac{1}{2}\zeta^2(s) - \frac{1}{2}\zeta(2s),\\ \zeta(\{s,s,s\}) &= \frac{1}{6}\zeta^3(s) - \frac{1}{2}\zeta(s)\zeta(2s) + \frac{1}{3}\zeta(3s),\ldots\end{aligned}$$

Crandall's recurrence is also a special case of Newton's formula

$$ke_k = \sum_{j=1}^{k}(-1)^{j+1}p_j e_{k-j}, \quad k \geq 0,$$

relating the Elementary Symmetric Functions e_k and and the Power-Sum Symmetric Functions p_r,

$$e_k = \sum_{j_1>\cdots>j_r} x_{j_1}\cdots x_{j_r}, \qquad p_r = \sum_{r>0} x_j^r,$$

with indeterminates $x_j = 1/j^s$, $e_r = \zeta(\{s\}^r)$ and $p_r = \zeta(rs)$.

Definition 4. Let $\vec{s} = (s_1,\ldots,s_k)$, $\vec{t} = (t_1,\ldots,t_r)$. The set $\texttt{stuffle}(\vec{s}|\vec{t})$ is defined as follows

1. $(s_1,\ldots,s_k,t_1,\ldots,t_r) \in \texttt{stuffle}(\vec{s}|\vec{t})$.
2. If (U,s_n,t_m,V) is in $\texttt{stuffle}(\vec{s}|\vec{t})$ then also are (U,t_m,s_n,V) and (U,s_n+t_m,V).

One also has

$$\#\texttt{stuffle}(\vec{s}|\vec{t}) = \sum_{j=0}^{r}\binom{k+j}{r}\binom{r}{j} = \sum_{j=0}^{\max(k,r)}\binom{k}{r}\binom{r}{j}2^j.$$

Theorem 6 (Stuffle Identities [4])**.**

$$\zeta(\vec{s})\zeta(\vec{t}) = \sum_{\vec{u}\in\mathtt{stuffle}(\vec{s}|\vec{t})} \zeta(\vec{u}).$$

For example

$$\zeta(r,s)\zeta(t) = \zeta(r,s,t) + \zeta(r,s+t) + \zeta(r,t,s) + \zeta(r+t,s) + \zeta(t,r,s).$$

4. Integral Representations for MDPs

Let $R_1, \ldots, R_k$ be disjoint sets of partitions of $\{1, \ldots, k\}$. For each $1 \le m \le n$, let

$$r_m = \sum_{i\in R_m} s_i \quad \text{and} \quad d_m = \prod_{i\in R_m} b_i.$$

From the gamma function identity

$$r^{-s}\Gamma(s) = \int_1^\infty (\log x)^{s-1} x^{-r-1} dx, \quad r, s > 0.$$

one gets

Proposition 1.

$$\lambda\binom{r_1, \ldots, r_n}{d_1, \ldots, d_n} = \left\{\prod_{j=1}^k \int_1^\infty \frac{(\log x_j)^{s_j-1}}{\Gamma(s_j)} \frac{dx_j}{x_j}\right\} \prod_{m=1}^n \left(d_m \prod_{j=1}^m \prod_{i\in R_j} x_i - 1\right)^{-1}.$$

For example, given a rational function on x and y, $R(x,y)$. Let $I(R)$ be the following *partition integrals*

$$I(R) = \int_1^\infty \int_1^\infty \frac{(\log x)^{s-1}(\log y)^{t-1}}{\Gamma(s)\Gamma(t)} \frac{dxdy}{xyR(x,y)}.$$

It follows that

$$\begin{aligned}
\lambda\binom{s+t}{ab} &= I(abxy-1),\\
\lambda\binom{s,t}{a,ab} &= I[(ax-1)(abxy-1)],\\
\lambda\binom{t,s}{b,ab} &= I[(by-1)(abxy-1)],\\
\lambda\binom{s}{a}\lambda\binom{t}{b} &= I[(ax-1)(by-1)].
\end{aligned}$$

From the rational identity

$$\frac{1}{(ax-1)(by-1)} = \frac{1}{abxy-1}\left(\frac{1}{ax-1} + \frac{1}{by-1} + 1\right),$$

one gets

$$\lambda\binom{s}{a}\lambda\binom{t}{b} = \lambda\binom{s,t}{a,ab} + \lambda\binom{t,s}{b,ab} + \lambda\binom{s+t}{ab}.$$

One can say that *stuffle identities are equivalent to rational identities via partition integrals.*

Definition 5. Given functions $f_j : [a,c] \to \mathbb{R}$ and the 1-forms $\Omega_j = f_j(y_j)dy_j$, the *iterated integral* over Ω_j are defined as follows

$$\int_a^c \Omega_1 \cdots \Omega_n = \begin{cases} 1 & \text{if } n = 0, \\ \int_a^c f(y_1) \int_a^{y_1} \Omega_2 \cdots \Omega_n \, dy_1 & \text{if } n > 0. \end{cases}$$

It turns out that MDPs have a convenient iterated integral representation in terms of 1-forms $\omega_b = dy/(y-b)$, i.e.

$$\lambda \binom{s_1, \dots, s_k}{b_1, \dots, b_k} = (-1)^k \int_0^1 \omega_0^{s_1 - 1} \omega_{b_1} \cdots \omega_0^{s_k - 1} \omega_{b_k}.$$

By the iterated integral representation, Broadhurst has generalized the notion of duality principle for MZVs to include the relations between iterated integrals involving the sixth root of unity using the change of variable $y \mapsto 1 - y$ at each level of integration [3]. This principle generates an involution $\omega_b \mapsto \omega_{1-b}$ holding for any complex value b. For example

$$\lambda \binom{2,1}{1,-1} = \int_0^1 \omega_0 \omega_1 \omega_{-1} = \int_0^1 \omega_2 \omega_0 \omega_1 = \lambda \binom{1,2}{2,1}$$

which is

$$\sum_{n \geq 1} \frac{1}{n^2} \left[\sum_{k=1}^{n-1} \frac{(-1)^k}{k} \right] = \sum_{n \geq 1} \frac{1}{n 2^n} \left[\sum_{k=1}^{n-1} \frac{2^k}{k^2} \right].$$

Several results can be similarly proved by using other transformations of variables in their integral representations. Here, the authors get

Theorem 7 (Cyclotomic). *Let n be a positive integer. Let $b_1, \dots, b_k$ be arbitrary complex numbers, and let $s_1, \dots, s_k$ be positive integers. Then*

$$\lambda \binom{s_1, \dots, s_k}{b_1^n, \dots, b_k^n} = n^{s-k} \sum_{\varepsilon_1, \dots, \varepsilon_k \in \{1, e^{2\pi i/n}, \dots, e^{2\pi(n-1)/n}\}} \lambda \binom{s_1, \dots, s_k}{\varepsilon_1 b_1, \dots, \varepsilon_k b_k}.$$

Theorem 8. *Let $s_1, \dots, s_k$ be nonnegative integers.*

$$\lambda \binom{1 + s_1, \dots, 1 + s_k}{-1, \dots, -1} = \sum \mu \left(\mathrm{Cat}_{j=1}^k \{-1\} \, \mathrm{Cat}_{i=1}^{s_j} \{\varepsilon_{i,j}\} \right) \prod_{j=1}^k \{-1\} \prod_{i=1}^{s_j} \varepsilon_{i,j},$$

where the sum is over all 2^s sequences of signs $(\varepsilon_{i,j})$ with each $\varepsilon_{i,j} \in \{1, -1\}$ for all $1 \leq i \leq s_j, 1 \leq j \leq k$, and Cat *denotes string concatenation.*

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