The tricritical scaling function of partially directed vesicles

Thomas Prellberg
University of Oslo

October 9, 1995

[summary by Helmut Prodinger]

This talk is largely based on [4]; some other "Prellbergs" are cited therein[1]. The author considers *staircase polygons.* They are defined as the set of all polygons on the square lattice whose perimeter consists of two fully directed walks with common start and end points.

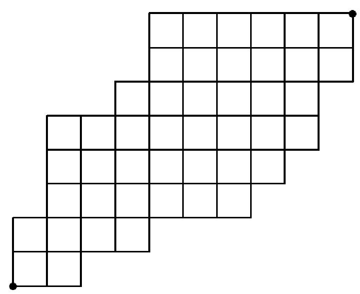


FIGURE 1. A staircase polygon with width 10, height 8, and area 45

If $c_m^{n_x,n_y}$ denotes the number of all staircase polygons with $2n_x$ horizontal and $2n_y$ vertical steps which enclose an area of size m, then the generating function

$$G(x,y,q)=\sum c_m^{n_x,n_y}x^{n_x}y^{n_y} \tag{1}$$

fulfills the functional equation

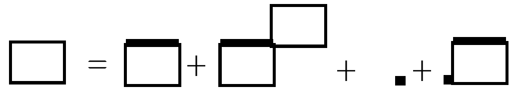
$$G(x,y,q)=\Big(G(qx,y,g)+qx\Big)\Big(G(x,y,g)+y\Big). \tag{2}$$

From this, an explicit expression is available;

$$G(x,y,q)=y\left(\frac{H(q^2x,qy,q)}{H(qx,qy,q)}-1\right)\quad\text{with}\quad H(x,y,q)=\sum_{n\ge0}\frac{(-x)^nq^{\binom{n}{2}}}{(q;q)_n(y;q)_n}, \tag{3}$$

where $(y;q)_n:=(1-y)(1-yq)(1-yq^2)\cdots(1-yq^{n-1})$.

[1]One might wonder why, then, the titles of talk and paper are so drastically different: "Vesicle" is a "closed fluctuating membrane", but combinatorialists think about polygons. And "tricritical" means that the generating function of interest has three ranges with a somehow different behaviour. The whole study is devoted to asymptotics of the generating function of interest, if the argument approaches the "tricritical" point.



$$G(x) \; = G(qx)y + G(qx)G(x) + \; qxy + qxG(x)$$

Prellberg derives this functional equation by setting up a *symbolic equation* which he translates into a functional equation for the generating function — very much in the tradition of the Algorithm seminar.

If we forget about the area, then we obtain the *perimeter generating function*

$$G(x,y,1) = \frac{1-x-y}{2} - \sqrt{\left(\frac{1-x-y}{2}\right)^2 - xy}. \tag{4}$$

The author concentrates in getting the following theorem.

THEOREM 1. *Set* $\epsilon = -\log q$. *Then, as* $q \to 1$,

$$G(x,y,q) \sim \frac{1-x-y}{2} + \sqrt{\left(\frac{1-x-y}{2}\right)^2 - xy}\left(\frac{\mathrm{Ai}'(\alpha\epsilon^{-2/3})}{\alpha^{1/2}\epsilon^{-1/3}\,\mathrm{Ai}(\alpha\epsilon^{-2/3})}\right). \tag{5}$$

Here, α *is some complicated function of* x *and* y *which simplifies to*

$$\alpha(x,y) \sim \left(\frac{4}{1-(x-y)^2}\right)^{4/3}\left(\left(\frac{1-x-y}{2}\right)^2 - xy\right) \tag{6}$$

for $(1-x-y)^2 \approx 4xy$. $\mathrm{Ai}(x)$ *is the* Airy function *(see* [5]*).*

Everything boils down to a study of the function $H(x,y,q)$, and the author comes up with a lemma.

LEMMA 1. *For* $x \in \mathbb{C}$, $|\arg(x)| < \pi$, $y \in \mathbb{C}$, $y \neq q^{-n}$ *for non-negative integers* n *and* $0 < q < 1$, *we have*

$$H(x,y,q) = \frac{(q;q)_\infty}{(y;q)_\infty}\frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty}\frac{(y/z;q)_\infty}{(z;q)_\infty}z^{-\log x/\log q}dz, \qquad 0 < \rho < 1. \tag{7}$$

Such a representation is no surprise at all; check out the wonderful survey papers [2] and [3]. The basic idea is to use the formula

$$\sum_{n\ge 0}(-x)^n c_n = \frac{1}{2\pi i}\int_{\mathcal{C}} x^s\, c(s)\frac{\pi}{\sin \pi s}\,ds \tag{8}$$

where $\mathcal{C}$ encloses the points $0, 1, \ldots$ in the counter-clock direction. The function $c(s)$ is an analytic continuation of the sequence c_n. Ramanujan was very fond of this formula, and it is also related with the names of Abel, Plana, and Lindelöf.

To do asymptotics, the author needs a better understanding of the 'ingredients' in his function $H(x,y,q)$ (a q-Bessel function), as $q \to 1$.

Interchanging sums,

$$\log(t;q)_\infty = -\sum_{m\ge 1}\frac{1}{m}\frac{t^m}{1-q^m}. \tag{9}$$

From here, *Euler's summation formula* gives for $|\arg(1-t)| < \pi$

$$\log(t;q)_\infty = \frac{1}{\log q}\operatorname{Li}_2(t) + \frac{1}{2}\log(1-t) + O(\log q), \tag{10}$$

with *Euler's dilogarithm*

$$\operatorname{Li}_2(t) = \sum_{m\ge 1}\frac{t^m}{m^2} = -\int_0^\infty \frac{\log(1-u)}{u}\,du. \tag{11}$$

For $(q;q)_\infty$ the author uses a *modular transformation*, viz. (see [1])

$$\log(q;q)_\infty = (r;q)^{1/24}\sqrt{\frac{2\pi}{-\log q}}\frac{1}{(r;r)_\infty} \tag{12}$$

to get

$$\log(q;q)_\infty = \frac{\pi^2}{6\log q} + \frac{1}{2}\log_{1/q}(2\pi) + O(\log q). \tag{13}$$

(The Mellin transform would also give this result.)

Continuing with approximations, the author notes the following.

LEMMA 2.

$$\begin{aligned} H(x,y,q) &= \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} \exp\left(\tfrac{1}{\epsilon}\left[\log(z)\log(x) + \operatorname{Li}_2(z) - \operatorname{Li}_2(y/z)\right]\right)\sqrt{\frac{1-y/z}{1-z}}dz \\ &\quad\times \exp\left(\tfrac{1}{\epsilon}(\operatorname{Li}_2(y) - \tfrac{\pi^2}{6})\right)\sqrt{\frac{2\pi}{\epsilon(1-y)}}\,(1+O(\epsilon)). \end{aligned} \tag{14}$$

The asymptotic evaluation of this integral will be done with the *saddle-point method.* There are two saddle points, and the whole thing becomes complicated when they coalesce (see [6] for an introduction to this problem).

A change of variable brings the function

$$V(\lambda) = \frac{1}{2\pi i}\int_{\mathcal{C}'} e^{u^3/3-\lambda u}\,du \tag{15}$$

into the picture ($\mathcal{C}'$ a certain contour). It is expressible by the Airy function $\operatorname{Ai}(\lambda)$.

Prellberg then presents his main lemma.

LEMMA 3. *Let* $0 < x, y < 1$ *and* $q = e^{-\epsilon}$. *Then*

$$\begin{aligned} H(x,y,q) &= \left(p_0\epsilon^{1/3}\operatorname{Ai}(\alpha\epsilon^{-2/3}) + q_0\epsilon^{2/3}\operatorname{Ai}'(\alpha\epsilon^{-2/3})\right) \\ &\quad\times \exp\left(\tfrac{1}{\epsilon}(\operatorname{Li}_2(y) - \tfrac{\pi^2}{6} + \log(x)\log(y)/2)\right)\sqrt{\frac{2\pi}{\epsilon(1-y)}}\,(1+O(\epsilon)), \end{aligned} \tag{16}$$

where

$$\frac{4}{3}\alpha^{3/2} = \log(x)\log\frac{z_m-\sqrt{d}}{z_m+\sqrt{d}} + 2\operatorname{Li}_2(z_m-\sqrt{d}) - 2\operatorname{Li}_2(z_m+\sqrt{d}) \tag{17}$$

with

$$z_{1,2} = z_m \pm \sqrt{d} \qquad z_m = \frac{1+y-x}{2} \quad \textit{and} \quad d = z_m^2 - y \tag{18}$$

and

$$p_0 = \left(\frac{\alpha}{d}\right)^{1/4}(1-x-y), \qquad q_0 = \left(\frac{d}{\alpha}\right)^{1/4}. \tag{19}$$

Bibliography

[1] Andrews (George E.). – *The Theory of Partitions.* – Addison-Wesley, 1976, *Encyclopedia of Mathematics and its Applications*, vol. 2.

[2] Flajolet (Philippe), Gourdon (Xavier), and Dumas (Philippe). – Mellin transforms and asymptotics: harmonic sums. *Theoretical Computer Science, Series A*, vol. 144, n° 1-2, June 1995, pp. 3–58. – Special Volume on Mathematical Analysis of Algorithms.

[3] Flajolet (Philippe) and Sedgewick (Robert). – Mellin transforms and asymptotics: finite differences and Rice's integrals. *Theoretical Computer Science*, vol. 144, n° 1-2, June 1995, pp. 101–124.

[4] Prellberg (Thomas). – Uniform q–series asymptotics for staircase polygons. *Journal of Physics Series A: Math. Gen.*, vol. 28, 1995, pp. 1289–1304.

[5] Whittaker (E. T.) and Watson (G. N.). – *A Course of Modern Analysis.* – Cambridge University Press, 1927, fourth edition. Reprinted 1973.

[6] Wong (Roderick). – *Asymptotic Approximations of Integrals.* – Academic Press, 1989.