# Some applications of the Mellin Transform in Signal processing

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[summary by Julie Bestel]

**Abstract**

The Mellin transform has been used in signal processing as a tool to investigate scale invariance. We review some of the recent studies by Wornell [3] and Cohen [2].

## 1. Introduction and examples

Assume we need to classify ships from radar signals [4]. The echo can be more or less compressed, depending on the angle between the axis of the ship and that of the radar signal. Nevertheless, one would like to be able to compare several echoes with different extension or compression rate, in order to decide whether or not they belong to the same kind of ship. A first approach would be to interpolate the signals, so that they would live on supports of equal size. A second one is to use some kind of transform that would ignore *scale variations*. The Mellin transform fulfils such a requirement; more precisely, the moduli of the Mellin transform of a signal $f(x)$ and of any dilation of $f(x)$ are the same. If time invariance is furthermore required, one may perform the *Fourier-Mellin transform*: Given an original real signal $f(x)$, the analytical signal is defined by $f_a(x) = f(x) + if_h(x)$, where $f_h(x)$ is the Hilbert transform of $f(x)$. Let $\mathcal{F}(f_a)(\omega) = F(\omega)$ be the Fourier transform of $f_a$. The quantity

$$\left|G_{|F|^2}(ix)\right|^2 = \left|\int_0^{+\infty} \omega^{ix-1}|F(\omega)|^2\, d\omega\right|^2$$

is both shift and scale invariant on the $x$ axis.

Section 2 gives a more detailed description of *scale invariant linear systems*. Section 3 presents a general framework for scale analysis.

## 2. Linear systems

If $x(t)$ is the input signal, a linear system outputs $y(t)$ as follows:

$$y(t) = S(x(t)) = \int_{-\infty}^{+\infty} x(\tau)K(t,\tau)\, d\tau$$

where $K(t,\tau)$ denotes the kernel of the system.

**2.1. Shift invariant systems.** As is well known, shift invariant systems are such that:

$$S(x(t-\tau)) = y(t-\tau) \Longleftrightarrow K(t,\tau) = V(t-\tau)$$

where $V$ is the impulse response of the system, i.e., $V(t) = S(\delta(t))$. It follows that $y$ is obtained by convolving $x$ and $V$:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)V(t-\tau)\,d\tau = (x \star V)(t).$$

The eigenfunctions of these systems are the exponential functions: $t \mapsto e^{st}$, $s \in \mathbb{C}$. The Laplace transform

$$\mathcal{L}(x)(s) = X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt$$

enables to change convolution into multiplication: $\mathcal{L}[(x \star y)](s) = X(s)Y(s)$.

**2.2. Scale invariant systems.** We are now interested in having $S(x(t/\tau)) = y(t/\tau)$. One can easily check that this is equivalent to $K(t,\tau) = aK(at, a\tau)$. The system $S$ is characterized by two *lagged* impulse responses:

$$\xi_+(t) = S(\delta(t-1)), \qquad \xi_-(t) = S(\delta(t+1))$$

$$y(t) = \int_0^{+\infty} x(\tau)\xi_+(t/\tau)\frac{d\tau}{\tau} - \int_0^{+\infty} x(-\tau)\xi_-(t/\tau)\frac{d\tau}{\tau}.$$

For causal signals and systems with causal response,

$$y(t) = \int_0^{+\infty} x(\tau)\xi_+(t/\tau)\frac{d\tau}{\tau} = (x \diamond \xi)(t) \qquad \text{(scale convolution)}.$$

The kernel $K$ is such that: $K(t,\tau) = \xi(t/\tau)/\tau$. The eigenfunctions of the operator thus defined are the functions $t \mapsto t^s$. The associated eigenvalue is the Mellin transform:

$$\mathcal{M}(x)(s) = M(s) = \int_0^{+\infty} \xi(\tau)\tau^{-s-1}\,d\tau.$$

We can then write: $\mathcal{M}[(x \diamond y)](s) = X(s)Y(s)$. The Mellin transform plays for scale convolution the role that the Laplace transform plays for ordinary convolution.

*Application to scale differential equations.* One defines the derivative with respect to the scale by:

$$\nabla_s(x)(t) = \lim_{\epsilon \to 1} \frac{x(\epsilon t) - x(t)}{\ln \epsilon}.$$

If $x$ is differentiable with respect to $t$, $\nabla_s(x)(t) = tx'(t)$. One can check that the derivative with respect to scale corresponds to a multiplication by $s$ in the Mellin domain.

**2.3. Generalized scale invariance.** More generally, one considers systems such that $S(x(t/\tau)) = \tau^\lambda y(t/\tau)$. This holds if and only if $K(t,\tau) = a^{-(\lambda-1)}K(at, a\tau)$. For causal signals, the *lagged impulse response* $\xi_+$ is such that:

$$y(t) = \int_0^{+\infty} x(\tau)\xi_+(t/\tau)\frac{d\tau}{\tau^{(1-\lambda)}}.$$

**2.4. Jointly time and scale invariant systems.** We now wish to have both

$$S(x(t-\tau)) = y(t-\tau) \qquad \text{and} \qquad S(x(t/\tau) = \tau^{\lambda} y(t/\tau).$$

One can show that the kernel should be a generalized homogeneous function of degree $\lambda - 1$:

$$v(t) = a^{-(\lambda-1)} v(at)$$

Hence,

$$v(t) = \begin{cases} C_1 |t|^{\lambda-1} u(t) + C_2 |t|^{\lambda-1} u(-t), & \text{if } -\lambda \notin \mathbb{N}, \\ C_1 |t|^{\lambda-1} u(t) + C_2 |t|^{\lambda-1} u(-t) + C_3 \delta^{(n)}(t), & \text{otherwise,} \end{cases}$$

where the $C_i$ are constants and $u(t)$ is the Heaviside function.

## 3. The scale representation

The starting point of this approach [2] is the following simple remark:

- The *content* of the signal $x$ at time $t$ is nothing but $x(t)$;
- the *content* of the signal $x$ at frequency $f$ is given by its Fourier transform $X(f)$.

Our purpose is then to define the concept of scale and the *content* of the signal $x$ at scale $c$. The idea consists in associating a *physical* quantity $a$ with an Hermitian operator $\mathcal{A}$. Let us begin with *common physical* quantities: time and frequency. The operators $T$ and $F$ respectively associated with $t$ and $f$ are:

$$T : x(t) \mapsto tx(t), \qquad F : x(t) \mapsto -i\frac{dx}{dt}.$$

In the frequency domain, we obtain:

$$T : X(f) \mapsto i\frac{dX}{df}, \qquad F : X(f) \mapsto fX(f).$$

It should be noticed that $T$ and $F$ do not commute:

$$[T, F] = TF - FT = i.$$

This is the reason why we get an incertitude principle on $t$ and $f$. We now define the scale operator as follows:

$$\mathcal{C} = \frac{1}{2}(TF + FT).$$

The following relations justify this definition:

$$e^{i\sigma\mathcal{C}} x(t) = e^{\sigma/2} x(e^{\sigma} t), \qquad e^{i\sigma\mathcal{C}} X(f) = e^{-\sigma/2} x(e^{-\sigma} f).$$

Whereas

$$e^{i\tau F} x(t) = x(t+\tau), \qquad e^{i\theta T} X(f) = X(f-\theta),$$
$$[T, \mathcal{C}] = T\mathcal{C} - \mathcal{C}T = iT, \qquad [T, F] = F\mathcal{C} - \mathcal{C}F = -iF.$$

Therefore, there exists an incertitude relation between scale and time, or between scale and frequency:

$$\Delta c \Delta t \geq \frac{1}{2} |\langle t \rangle|$$

where the average time is defined by

$$\langle t \rangle = \int t |x(t)|^2 \, dt.$$

The equality is reached for the signal:

$$x(t) = kt^{\alpha} \exp\left[-\beta t + i\langle c\rangle \ln\left(\frac{t}{\langle t\rangle}\right)\right].$$

Dually, we get $\Delta f \Delta c \geq \frac{1}{2}|\langle f\rangle|$.

Let $\gamma(c,t)$ be the eigenfunctions of $\mathcal{C}$: $\mathcal{C}\gamma(c,t) = c\gamma(c,t)$. We find, for $t \geq 0$:

$$\gamma(c,t) = \frac{1}{\sqrt{2\pi}} t^{ic-\frac{1}{2}}.$$

We can now produce the direct and inverse transforms, for $t \geq 0$:

$$D(c) = \int x(t)\gamma^*(c,t)\,dt = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} x(t)t^{-ic-\frac{1}{2}}\,dt,$$

$$x(t) = \int D(c)\gamma(c,t)\,dc = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} D(c)t^{ic-\frac{1}{2}}\,dc.$$

One can notice that we have recovered a Mellin transform, in the special case when $\Re(s) = \frac{1}{2}$. That is why the Mellin transform was commonly renamed *Scale transform* in signal processing.

The average scale of a signal is given by: $\langle c\rangle = \int c|D(c)|^2\,dc$. One obtains

$$\langle c\rangle = \int_0^{+\infty} t\phi'(t)|x(t)|^2\,dt = \int_{-\infty}^{+\infty} f\psi'(f)|X(f)|^2\,df.$$

One can deduce from these relations a notion of *instantaneous scale*, at time $t$: $c_t = t\phi'(t)$, and at frequency $f$: $c_f = -f\psi'(f)$.

A more unified presentation can be found in [1, 2].

## Bibliography


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