Asymptotics and scalings for large product-form networks via the Central limit theorem

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[summary by Philippe Robert]

1. Introduction

This talk considers the following closed queueing networks: there are n queues and m_n customers traveling in the network, the service rate at queue k when there are q_k customers is $\mu_{k,n}(q_k)$. A customer finishing his service at queue k goes to queue l with probability $p_{k,l}$ where $P_n = (p_{k,l})$ is an irreducible stochastic matrix with invariant measure $\pi_n = (\pi_{1,n}, \ldots, \pi_{n,n})$, defined by $\pi_n P_n = \pi_n$ and $\pi_{1,n} + \cdots + \pi_{n,n} = 1$. The service discipline can be FIFO, LIFO, or Processor sharing. To this network is associated a Markov process given by the vector of the number of customers in the queues $(Q_{k,n})$. It is well known that this Markov process has a unique equilibrium measure P_n such that, if $q_1, \ldots, q_n \geq 0$ and $q_1 + \cdots + q_n = m_n$,

$$P_n(Q_{1,n} = q_1, \ldots, Q_{n,n} = q_n) = Z_{m_n,n}^{-1} \prod_{k=1}^{n} \frac{\pi_{k,n}^{q_k}}{\mu_{k,n}(1) \cdots \mu_{k,n}(q_k)},$$

with the normalizing condition

$$Z_{m,n} = \sum_{q_1+\cdots+q_n=m} \prod_{k=1}^{n} \frac{\pi_{k,n}^{q_k}}{\mu_{k,n}(1) \cdots \mu_{k,n}(q_k)}.$$

The explicit expression for the equilibrium measure is not really informative because of the normalizing constant which is not easy to handle. It is difficult to get a qualitative insight on the network (such as the mean queue lengths and their variances). A way to cope with this problem is to consider asymptotics. The paper considers the case where the number of queues and the number of customers tend to infinity with some normalization between them.

2. The equivalent network

The main idea is to introduce the open network defined by n independent parallel queues with service rate $\mu_{k,n}(x)$ and arrival intensity $\lambda_n \pi_{k,n}$ at queue k.

The distribution of the number $X_{k,n}$ of clients in queue k is given by

$$P(X_{k,n} = x) = \frac{1}{f_{k,n}} \frac{(\lambda_n \pi_{k,n})^x}{\mu_{k,n}(1) \cdots \mu_{k,n}(x)}$$

where $f_{k,n}$ is a (simple) normalizing constant.

THEOREM 1. *For any choice of m_n, there exists a unique λ_n such that if $S_n = X_{1,n} + \cdots + X_{n,n}$, then $E(S_n) = m_n$. In this case for any $q_1, \ldots, q_n \geq 0$ and $1 \leq \ell \leq n$,*

$$P(Q_{1,n} = q_1, \ldots, Q_{n,n} = q_n) = \frac{1}{P(S_n = m_n)} \prod_{k=1}^{n} P(X_{k,n} = q_k),$$

$$P(Q_{1,n} = q_1, \ldots, Q_{\ell,n} = q_\ell) = \prod_{k=1}^{\ell} P(X_{k,n} = q_k) \frac{P(\sum_{k=\ell+1}^{n} X_{k,n} \sum_{k=\ell+1}^{n} m_{k,n})}{P(S_n = m_n)}$$

$$\times P(X_{1,n} = q_1, \ldots, X_{\ell,n} = q_\ell | S_n = m_n).$$

Starting from this representation, the asymptotic results concerning the network are proved via asymptotic results on S_n. Basically, in the same way as Kolchin [2] in another context, the authors use local limit theorems of the following form.

THEOREM 2. *Under "suitable" conditions, there exists a distribution with density h and a sequence a_n such that, for any integer x, $\lim_{n\to\infty} a_n P(S_n - m_n = x) - h(x/a_n) = 0$.*

3. Asymptotic expansions

The queues are partitioned into two sets, F_n and I_n. The set F_n contains those queues k for which $\liminf_{q\to\infty} \sqrt[q]{\mu_{k,n}(1) \cdots \mu_{k,n}(q)} < \infty$; the set I_n contains the other ones.

DEFINITION 1. Let

$$\mu_{k,n} = \begin{cases} \liminf_{q\to\infty} \sqrt[q]{\mu_{k,n}(1) \cdots \mu_{k,n}(q)}, & \text{if } k \in F_n, \\ \mu_{k,n}(1), & \text{if } k \in I_n, \end{cases} \qquad \text{and} \qquad \lambda_n^0 = \min_{k \in F_n} \frac{\mu_{k,n}}{\pi_{k,n}}.$$

A sequence m_n^0 is said to be *weakly critical*, if for any $0 < t < 1$, $g(t) = \limsup_{n\to\infty} m_n(t\lambda_n^0)/m_n^0$ exists and $\lim_{t\to 1-} g(t)$ be either 1 or ∞.

The *critical sequences* m_n^0 allow to distinguish between saturated and non-saturated regimes of the network, depending on the limit of $g(t)$ at 1. One of the main results on the asymptotic expansion of the equilibrium measure is the following theorem.

THEOREM 3. *Assume* $\lim_{n\to\infty} \max_{1\leq k\leq n} (\pi_{k,n}/\mu_{k,n})/[\pi_{1,n}/\mu_{1,n} + \cdots + \pi_{n,n}/\mu_{n,n}] = 0$. *Let m_n^0 be a weakly critical sequence, with the associated function $g(t)$. Assume that* $\lim_{t\to 1-} g(t) = 1$. *If moreover,* $\limsup_{n\to\infty} m_n/m_n^0 < 1$ *then, for any finite index j,*

$$P(Q_{1,n} = q_1, \ldots, Q_{j,n} = q_j) = \prod_{k=1}^{j} P(X_{k,n} = q_k) \left[1 + O\left(\frac{1}{m_n}\right)\right].$$

In particular, for all $k \in \{1, \ldots, n\}$, $E(Q_{k,n})$ is uniformly bounded in n.

Bibliography

[1] Fayolle (Guy) and Lasgouttes (Jean-Marc). – *Asymptotics and Scalings for Large Closed Product-form Networks via the Central Limit Theorem.* – Technical Report n° 2754, Institut National de Recherche en Informatique et en Automatique, 1996.

[2] Kolchin (V. F.). – *Random Mappings.* – Optimization Software, New York, 1986. Translated from *Slučajnye Otobraženija*, Nauka, Moscow, 1984.