Measures of distinctness for summands in partitions and compositions

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[summary by Philippe Dumas]

Abstract

Statistical properties of integer partitions and compositions are studied. The approach is based on generating functions and complex analysis, and uses Mellin transform.

The problem under treatment is mainly based on a work by Richmond and Knopfmacher [4], who considered compositions with distinct summands. It is also based on a work by Knopfmacher and Mays [2], who studied the number and the sum of distinct summands in compositions by elementary means. The approach of Hwang and Yeh [1] is different. It is based on generating functions and complex analysis, which allows them to consider a general scheme: the summands are taken form an infinite positive integer sequence (λ_j), and various types of partitions or compositions, inspired from combinatorial data structures, are studied.

There are different ways to estimate the degree of distinctness between the summands of a partition or of a composition. In this summary we content ourselves with the number of summands which occur h times or more in a partition or composition, though Hwang and Yeh consider many other criteria. This number may be viewed as a random variable $X_n^{[h]}$ indexed by the sum n of the partition or composition. In the case of compositions, the formula

$$\sum_{n\geq 1} c_n \,\mathrm{E}(X_n^{[h]}) = \sum_{j\geq 1} \frac{z^{h\lambda_j}}{(1-\Lambda(z))(1-\Lambda(z)+z^{\lambda_j})^h},$$

where c_n is the number of compositions of n and $\Lambda(z) = \sum_j z^{\lambda_j}$, provides a way to determine the asymptotic behavior of the mean $\mathrm{E}(X_n^{[h]})$.

Let us consider the simple case $\lambda_j = j$; so that $c_n = 2^{n-1}$. We have

$$\mathrm{E}(X_n^{[1]}) = \log_2 n - \frac{3}{2} + \frac{\gamma}{\log 2} - \frac{1}{2}\sum_{k\neq 0} \Gamma(\chi_k) n^{-\chi_k} + O\left(\frac{\log n}{n}\right),$$

with $\chi_k = 2ik\pi/n$. The proof is in four steps and relies on the formula

$$\mathrm{E}(X_n^{[1]}) = \sum_{j=1}^{n}\left(1 - \frac{1}{2^{n-1}}[z^n]\frac{1-z}{1-2z+z^j(1-z)}\right).$$

First, Rouché's theorem implies that the polynomial $1-2z+z^j(1-z)$ has only one root $(1+\varepsilon_j)/2$ inside the unit circle. Next the Lagrange inversion theorem gives an explicit expression of ε_j, namely

$$\varepsilon_j = \sum_{\ell\geq 1} \frac{1}{2^{(j+1)\ell}\ell} \sum_{i=0}^{\ell-1} \binom{\ell}{i+1} (-1)^{\ell-i-1} \binom{k\ell}{i}.$$

The singularities of the generating function being known, the next stage is an application of Cauchy's formula. One obtains

$$\mathrm{E}(X_n^{[1]}) = \sum_{j=1}^{n} \left(1 - \frac{1}{1+\varepsilon_j^n}\right) + O\left(\frac{\log n}{n}\right).$$

This new sum is a harmonic sum which can be expressed as an inverse Mellin transform, hence

$$\mathrm{E}(X_n^{[1]}) = \frac{-1}{2i\pi} \int_{-1/2-i\infty}^{-1/2+i\infty} \Gamma(s) n^{-s} U(s)\, ds + O\left(\frac{\log n}{n}\right),$$

with

$$U(s) = \begin{cases} \sum_{j\geq 1} \log(1+\varepsilon_j)^{-s}, & \Re(s) < 0, \\ 4^s/(1-2^s) + V(s), & \Re(s) < 1. \end{cases}$$

This provides the announced formula. The analysis differs from Knuth's one [3] and gives a better error term. The big oh term may be replaced by a sum

$$\sum_{k\geq 1} \frac{1}{n^k} \sum_{\ell=0}^{k} \varpi_{k,\ell}(\log_2 n) \log^\ell n,$$

where the $\varpi_{k,\ell}$s are periodic functions. More generally one obtains

$$\mathrm{E}(X_n^{[h]}) = \mathrm{E}(X_n^{[1]}) - \sum_{j=1}^{h-1} \frac{1}{j! \log 2} \left(1 + \sum_{k\neq 0} \Gamma(j+\chi_k) n^{-\chi_k}\right) + O\left(\frac{\log n}{n}\right).$$

All this is relative to the case $\lambda_j = j$.

In the general case $\Lambda(z) = \sum_j z^{\lambda_j}$ cannot be written as $z^a \Lambda_1(z^d)$ with $d \geq 2$ and the count function

$$A(x) = \sum_{\lambda_j \leq x} 1$$

tends to infinity with x. Under these conditions, one obtains

$$\mathrm{E}(X_n^{[h]}) = A(\log_r(cn)) + O(1),$$

where r and c are defined by $\Lambda(\rho) = 1$, $r = 1/\rho$ end $c = 1/\rho/\Lambda'(\rho)$. One may say there is a logarithmic transition from the behavior of A to the behavior of $\mathrm{E}(X_n^{[h]})$.

Hwang and Yeh consider others compositions like cyclic compositions where compositions are considered up to circular permutation, or branching compositions where the summands label the nodes of a binary tree.

Bibliography

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