Computing the Distance of a Point to an Algebraic Hypersurface and Application to Exclusion Methods

Xavier Gourdon

Algorithms Project, INRIA Rocquencourt

February 12, 1996

[summary by Pierre Nicodème]

Abstract

We compute lower bounds for the distance in $\mathbb{C}^n$ from a point u to an algebraic surface $\mathcal{Z}$. Such lower bounds or proximity tests give an approximation of $\mathcal{Z}$. We present tests based on both Taylor's formula and a generalization of the Dandelin-Graeffe process to the multivariate case, and their application to the exclusion method [2].

1. Introduction

Given a point a in $\mathbb{C}^n$, and an algebraic hypersurface

$$\mathcal{Z}(P) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \,|\, P(z_1, \ldots, z_n) = 0\},$$

with $P \in \mathbb{C}[z_1, \ldots, z_n]$, we want to evaluate the distance $d(a, \mathcal{Z})$ corresponding to the norm

$$\|z\| = \max_{1 \le k \le n} |z_i|.$$

By shifting the variable z, we can restrict to the case $a = 0$.

2. Univariate Polynomials

Let $P(z) = \sum_{i=0}^{d} a_i z^i \in \mathbb{C}[z]\,, a_d \neq 0$, and $\mathcal{Z}(P) = \{U_1, \ldots, U_d\}$. We want to evaluate $d(0, \mathcal{Z}) = \min_i |U_i|$. In Henrici [4, vol. 1], Theorems 6.4.d and 6.4.i give the following classical bound for $\mathcal{Z}(P)$:

Proposition 1. *If $\rho(P)$ is the nonnegative root of the equation $|a_0| = \sum_{j=1}^{d} |a_j| \rho^j$, then*

$$\rho(P) \le d(0, \mathcal{Z}) \le \frac{1}{2^{1/d} - 1} \rho(P) \approx \frac{d}{\log 2} \rho(P).$$

Graeffe Iteration. With $P(z) = a_d \prod_{i=1}^{d} (z - U_i)$, we consider

$$P(z)P(-z) = (-1)^d a_d^2 \prod_{i=1}^{d} (z^2 - U_i^2) = P^{\langle 1 \rangle}(z^2).$$

We note $P^{\langle 1 \rangle}$ the classical Graeffe iterate; the roots of $P^{\langle 1 \rangle}$ are the squares of those of P, and $d(0, \mathcal{Z}(P^{\langle 1 \rangle})) = d(0, \mathcal{Z}(P))^2$; we have

$$\rho(P^{\langle 1 \rangle}) \le d(0, \mathcal{Z}(P^{\langle 1 \rangle})) \le \frac{\rho(P^{\langle 1 \rangle})}{2^{1/d} - 1};$$

so with $\rho_1 = \sqrt{\rho(P^{\langle 1\rangle})}$, we get

$$\rho_1 \le d(0, \mathcal{Z}(P)) \le \frac{\rho_1}{(2^{1/d}-1)^{1/2}}.$$

Generally, we define $P^{\langle k\rangle} = \text{Graeffe}(P^{\langle k-1\rangle})$; then, we get $d(0, \mathcal{Z}(P^{\langle k\rangle})) = d(0, \mathcal{Z}(P))^{2^k}$; with $\rho_k = \rho(P^{\langle k\rangle})^{1/2^k}$, we have

$$\rho_k \le d(0, \mathcal{Z}(P)) \le \frac{\rho_k}{(2^{1/d}-1)^{1/2^k}}.$$

The upper bound tends rapidly to the lower bound as k increases, thus we have obtained an effective process to compute $d(0, \mathcal{Z})$.

Computing the $P^{\langle k\rangle}$. With $A(z) = \sum_{i\equiv 0 \bmod 2} a_i z^{i/2}$ and $B(z) = \sum_{i\equiv 1 \bmod 2} a_i z^{(i-1)/2}$, we have

$$P(z)P(-z) = A(z^2)^2 - z^2 B(z^2)^2,$$

and therefore,

$$\text{Graeffe}(P) = A(z)^2 - zB(z)^2.$$

A practical problem is that the coefficient size doubles at each Graeffe iteration.

3. Multivariate Polynomials

In the multivariate case, the polynomial $P(z)P(-z)$ can not be written as $Q(z^2)$ where $Q(z)$ is a polynomial, thus we need to modify the definition. We generalize the Graeffe process to the multivariate case as follows:

Definition 1. We call the N-th Graeffe iterate of $P(z) \in \mathbb{C}[z_1, \ldots, z_n]$ the polynomial $P^{[N]}(z)$ defined by

$$P^{[N]}(z) = \prod_{j=0}^{2^N-1} P(\omega^j z), \qquad \omega = \exp\left(\frac{2i\pi}{2^N}\right), \quad i^2 = -1,$$

where $\omega^j z$ denotes the point $(\omega^j z_1, \ldots, \omega^j z_n)$.

Proposition 2. *For all non negative integer N, the N-th Graeffe iterate of $P(z)$ writes as*

$$P^{[N]}(z) = \sum_{j\ge 0} B_j^{[N]}(z),$$

where the $B_j^{[N]}$'s are homogeneous polynomials of degree $2^N j$. The $(N+1)$-st Graeffe iterate can be computed from the N-th thanks to the formula

$$P^{[N+1]}(z) = P_0^{[N]}(z)^2 - P_1^{[N]}(z)^2, \qquad P_k^{[N]}(z) = \sum_{j\equiv k \bmod 2} B_j^{[N]}(z).$$

With the multivariate Graeffe process, we easily generalize the univariate algorithm to compute $d(0, \mathcal{Z})$ in the multivariate case.

Theorem 1. *Let $P(z)$ be a polynomial in $\mathbb{C}[z_1, \ldots, z_n]$ of total degree d. Let $P^{[N]}(z) = \sum_{j\ge 0} B_j^{[N]}(z)$ be its N-th Graeffe iterate and R_N the non-negative solution of the equation in R*

$$|P^{[N]}(0)| = \sum_{j\ge 1} \|B_j^{[N]}\|_\infty R^j, \tag{1}$$

d	r_0/d	r_1/d	r_2/d	r_3/d	r_4/d
2	0.7673	0.9725	0.9996	1.0000	1.0000
5	0.6525	0.9479	0.9973	1.0000	1.0000
7	0.6325	0.9400	0.9960	0.9999	1.0000
15	0.6067	0.9271	0.9938	0.9999	1.0000

TABLE 1. Some values of $r_N/d(0, \mathcal{Z}_{n,d})$ for $n = 2$.

d	r_0/d	r_1/d	r_2/d	r_3/d
2	0.5832	0.6338	0.8108	0.8224
3	0.4802	0.5108	0.6478	0.7561

TABLE 2. Some values of $r_N/d(0, \mathcal{Z}_{n,d})$ for $n = 7$.

where $\|B_j^{[N]}\|_\infty = \sup_{\|z\|=1} \|B_j(z)\|$. *Then we have*

$$r_N \leq d(0, \mathcal{Z}) \leq \left(\frac{1}{2^{1/d} - 1}\right)^{2^{-N}} r_N, \qquad r_N = R_N^{2^{-N}}. \tag{2}$$

Computing $\|B_j^{[N]}\|_\infty$ raises a difficult practical problem; therefore, we make use of the norm $\|\sum_\alpha a_\alpha z^\alpha\| = \sum |a_\alpha|$, easy to compute. Our main result is stated using this norm; one demonstrates the equivalence of the norms $\|\cdot\|_\infty$ and $\|\cdot\|$ by combination of the Parseval identity and of the Cauchy-Schwarz inequality.

THEOREM 2. *Let ρ_N be the unique nonnegative solution of*

$$|P^{[N]}(0)| = \sum_{j=1}^{d} \|B_j^{[N]}\| \rho^j \tag{3}$$

The distance from 0 to $\mathcal{Z}$ satisfies

$$r_N \leq d(0, \mathcal{Z}) \leq \kappa_N r_N, \tag{4}$$

where

$$r_N = \rho_N^{2^{-N}} \quad \text{and} \quad \kappa_N = \left(\frac{1}{2^{1/d} - 1}\sqrt{\binom{2^N + n - 1}{n - 1}}\right)^{1/2^N}.$$

Moreover $\lim_{N\to\infty} \kappa_N = 1$, *which implies* $\lim_{N\to\infty} r_N = d(0, \mathcal{Z})$.

4. Examples

We take a polynomial of degree d in n variables: $P_{n,d} = \sum_{j=1}^{n} (1 - z_j)^d - 1$. With $\mathcal{Z}_{n,d} = \mathcal{Z}(P_{n,d})$, we have $d(0, \mathcal{Z}_{n,d}) = 1 - \frac{1}{n^{1/d}}$.

Tables 1 and 2 give the value of the ratio $r_N/d(0, \mathcal{Z}_{n,d})$ of Theorem 3 for several values of n, d and N. The computations were performed in Maple. These examples show that the bound is quite good for a small value N of Graeffe iterates.

5. Exclusion methods

We give the principle of the method for a polynomial of one variable $P(z) \in \mathbb{C}[z]$.

- Let the *exclusion function* be: $z_0 \mapsto \rho(z_0)$, with ρ given by theorem 2 after a proper shift of the variable, and
  (1) $\rho(z_0) = 0 \iff P(z_0) = 0$,
  (2) P has no zero in $|z - z_0| < \rho(z_0)$, which is equivalent to $\rho(z_0) \leq d(z_0, \mathcal{Z})$;
- then, the *exclusion test* is: let C be a square of centre z_0 and half-side $a > 0$. If $\rho(z_0) \geq \sqrt{2}a$, C contains no zero of P.

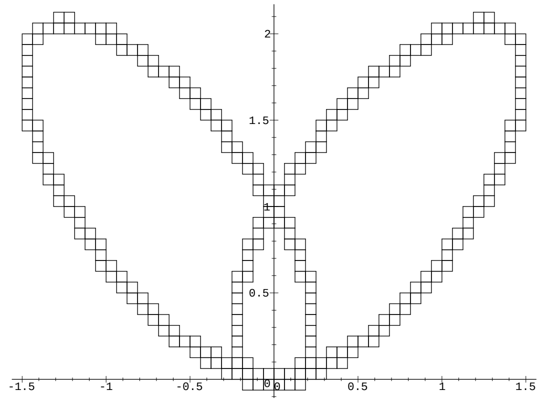

FIGURE 1. Representing by exclusion the curve $y^4-2y^3+y^2-3x^2y+2x^4=0$ (petal).

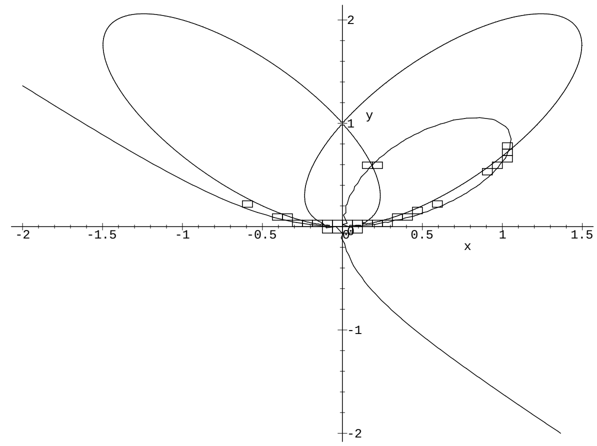

FIGURE 2. Intersection of the curves $x^3+y^3-2xy=0$ (Descartes folium) and $y^4-2y^3+y^2-3x^2y+2x^4=0$ (petal).

Exclusion algorithm.

- Consider the reciprocal polynomial $R(z)$ of $P(z)$; compute by Graeffe a lower bound of the smallest root of $R(z)$, which gives an upper bound b_u of the largest root of $P(z)$;
- Start from a big square centred at the origin, with side $2b_u$, which contains all the roots of $P(z)$;
- Recursively split the square in four squares of equal size, discarding by the exclusion test squares containing no zeros;
- Stop the recursion when the desired precision is reached (the surface of the area covering the zeros decreases exponentially fast to zero).

Figure 1 shows an application of the exclusion method to localize an algebraic curve in $\mathbb{R}^2$.

For an algebraic variety $\mathcal{Z}_i = \mathcal{Z}(P_i)$ and $\mathcal{Z} = \bigcap_i \mathcal{Z}(P_i)$, with $P_1, \ldots, P_m \in \mathbb{C}[z_1, \ldots, z_n]$, let $\rho_i(z_0)$ be an exclusion function defined by theorem 2 for $P_i, (1 \leq i \leq m)$; we can define an exclusion function for the variety as $\rho(z_0) = \sup_{1 \leq i \leq m} \rho_i(z_0)$.

An application of exclusion method to localize the intersection of two curves in $\mathbb{R}^2$ is given in Figure 2.

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