Euler sums

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January 29, 1996

[summary by Jean-Paul Allouche]

In 1742 Goldbach wrote a letter to Euler proposing the study of the sums

$$S_{p,q} := \sum_{n=1}^{\infty} \left(\frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p} \right) \frac{1}{n^q} = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q},$$

where $H_n^{(r)}$ and $H_n = H_n^{(1)}$ are the *harmonic numbers* defined by

$$H_n^{(r)} := \sum_{j=1}^{n} \frac{1}{j^r}.$$

Euler was able to compute all the sums $S_{p,q}$ for $p + q \leq 13$, for example

$$\sum_{n=1}^{\infty} \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} \right) \frac{1}{n^2} = 2\zeta(3).$$

Then, in 1906, Nielsen gave relations linking the sums $S_{p,q}$ having the same weight $w = p + q$. Hence the $S_{p,q}$ of odd weight are polynomials in the values of zeta, for example

$$S_{2,5} = \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^5} = 5\zeta(2)\zeta(5) + 2\zeta(3)\zeta(4) - 10\zeta(7).$$

Many similar identities have then been found or conjectured. Some of them involve multiple zeta functions; see the papers of Bayley, D. and J. Borwein, De Doelder, Don Zagier, Girgensohn, Hoffman, Markett.

The authors [1] propose a simple and unifying method that gives most of the known results about these identities. Furthermore they are able to prove some conjectures. The key idea is to use a contour integral *with a well-chosen kernel.*

1. The idea of the authors: a simple case

Let us denote by $I(p, q)$ the integral

$$I(p,q) = \frac{1}{2i\pi} \int_C (\psi(-s) + \gamma)^2 \frac{ds}{s^q},$$

where C is a circle whose radius goes to infinity, and where ψ is the logarithmic derivative of the Γ function. Denoting by γ the Euler constant, we have

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z} \right).$$

Hence, when s tends to a positive integer m, then

$$(\psi(-s)+\gamma)^2 \underset{s\to m}{=} \frac{1}{(s-m)^2} + 2H_m \frac{1}{s-m} + \cdots .$$

If s tends to 0, we use the relation $\psi(s)+\gamma = -1/s + \zeta(2)s - \zeta(3)s^2 + \cdots$. Hence, by residue computation the Euler sum $S_{1,q}$ can be expressed as an explicit quantity which is "homogeneous" of degree 2 in the zeta values.

In the general case the authors consider integrals

$$\frac{1}{2i\pi}\int_C r(s)\xi(s)\,ds,$$

where r is a rational function, and ξ a *suitable* kernel. They then obtain numerous results: some of them were already known, but some of them were only conjectures.

2. A zoo of beautiful identities

The authors obtain the following results.

THEOREM 1 (EULER). *Let q be an integer ≥ 2. Then*

$$S_{1,q} = \sum_{n=1}^{\infty} \frac{H_n}{n^q} = \left(1+\frac{1}{2}q\right)\zeta(q+1) - \frac{1}{2}\sum_{k=1}^{q-2}\zeta(k+1)\zeta(q-k).$$

For example

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2} = 2\zeta(3), \quad \sum_{n=1}^{\infty}\frac{H_n}{n^3} = \frac{5}{4}\zeta(4), \quad \sum_{n=1}^{\infty}\frac{H_n}{n^4} = 3\zeta(5) - \zeta(2)\zeta(3).$$

THEOREM 2 (EULER, BORWEIN ET AL.). *If the weight $m = p+q$ is odd, then*

$$\sum_{n=1}^{\infty}\frac{H^{(p)}(n)}{n^q} = \zeta(m)\left[\frac{1}{2} - \frac{(-1)^p}{2}\binom{m-1}{p} - \frac{(-1)^p}{2}\binom{m-1}{q}\right] + \frac{1-(-1)^p}{2}\zeta(p)\zeta(q)$$
$$+ (-1)^p \sum_{k=1}^{\lfloor \frac{p}{2}\rfloor}\binom{m-2k-1}{q-1}\zeta(2k)\zeta(m-2k) + (-1)^p\sum_{k=1}^{\lfloor \frac{q}{2}\rfloor}\binom{m-2k-1}{p-1}\zeta(2k)\zeta(m-2k),$$

where any occurrence of $\zeta(1)$ has to be replaced by 0.

If we then use the symmetry $S_{p,q} + S_{q,p} = \zeta(p)\zeta(q) + \zeta(p+q)$, we obtain

$$5\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^6} + 2\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^5} = -\frac{21}{2}\zeta(8) + 10\zeta(3)\zeta(5) + \frac{9}{2}\zeta(4)^2,$$
$$7\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^8} + 2\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^7} = -33\zeta(10) + 14\zeta(3)\zeta(7) + 15\zeta(4)\zeta(6) + 8\zeta(5)^2,$$
$$7\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^8} - 2\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^6} = -\frac{229}{5}\zeta(10) + 14\zeta(3)\zeta(7) + 21\zeta(4)\zeta(6) + 10\zeta(5)^2,$$
$$9\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^{10}} + 2\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^9} = -\frac{143}{2}\zeta(12) + 18\zeta(3)\zeta(9) + 21\zeta(4)\zeta(8) + 24\zeta(5)\zeta(7) + \frac{25}{2}\zeta(6)^2,$$

$$8\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^{10}}-\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^8}=-\frac{575}{7}\zeta(12)+16\zeta(3)\zeta(9)+24\zeta(4)\zeta(8)+28\zeta(5)\zeta(7)+\frac{295}{21}\zeta(6)^2,$$

$$7\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^{10}}+\sum_{n=1}^{\infty}\frac{H_n^{(5)}}{n^7}=-73\zeta(12)+28\zeta(5)\zeta(7)+21\zeta(4)\zeta(8)+14\zeta(3)\zeta(9)+\frac{35}{3}\zeta(6)^2.$$

Then

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}=\frac{7}{4}\zeta(4),\quad \sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}=\frac{1}{2}\zeta(3)^2+\frac{1}{2}\zeta(6),\quad \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}=\zeta(3)^2-\frac{1}{3}\zeta(6).$$

Theorem 3 (Borwein et al.). *The following relations hold.*

$$\sum_{n=1}^{\infty}\frac{(H_n)^2}{n^q}-S_{2,q}=qS_{1,q+1}-\frac{q(q+1)}{6}\zeta(q+2)+\zeta(2)\zeta(q).$$

For example

$$\sum_{n=1}^{\infty}\frac{(H_n)^2}{n^3}=\frac{7}{2}\zeta(5)-\zeta(2)\zeta(3),$$

$$\sum_{n=1}^{\infty}\frac{(H_n)^2}{n^5}=6\zeta(7)-\zeta(2)\zeta(5)-\frac{5}{2}\zeta(3)\zeta(4),$$

$$\sum_{n=1}^{\infty}\frac{(H_n)^2}{n^7}=\frac{55}{6}\zeta(9)-\zeta(2)\zeta(7)-\frac{7}{2}\zeta(3)\zeta(6)-\frac{5}{2}\zeta(4)\zeta(5)+\frac{1}{3}\zeta(3)^3,$$

and

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^6}-\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^6}=\frac{91}{12}\zeta(8)-8\zeta(3)\zeta(5)+\zeta(2)\zeta(3)^2,$$

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^8}-\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^8}=\frac{473}{40}\zeta(10)-10\zeta(3)\zeta(7)-5\zeta(5)^2+\zeta(4)\zeta(3)^2+2\zeta(2)\zeta(3)\zeta(5),$$

$$\sum_{n=1}^{\infty}\frac{(H_n)^2}{n^2}=\frac{17}{4}\zeta(4),$$

$$\sum_{n=1}^{\infty}\frac{(H_n)^2}{n^4}=\frac{307}{24}\zeta(6)-5\zeta(2)\zeta(4)-2\zeta(3)^2.$$

Theorem 4. *If $i+j+k$ is odd, with $i>1$, $j>1$, $k>1$, then*

$$[(-1)^k+(-1)^{i+j}]\sum_{n\geq 1}\frac{H_n^{(i)}H_n^{(j)}}{n^k}+A+B+C+D+E+F=0,$$

where

$$A=(-1)^{i+j+k}\zeta(i)\zeta(j)\zeta(k)+(-1)^{i+k}\zeta(i)S_{j,k}+(-1)^{j+k}\zeta(j)S_{i,k},$$

$$B=-2(-1)^k\sum_{q+2r+t=i}\binom{j+q-1}{q}\binom{k+t-1}{k-1}[(-1)^qS_{j+q,k+t}+(-1)^j\zeta(j+q)\zeta(k+t)]\zeta(2r),$$

$$C=-2(-1)^k\sum_{p+2r+t=j}\binom{i+p-1}{p}\binom{k+t-1}{k-1}[(-1)^pS_{i+p,k+t}+(-1)^i\zeta(i+p)\zeta(k+t)]\zeta(2r),$$

$$
\begin{aligned}
D &= -2(-1)^k \sum_{2r+t=i+j} \binom{k+t-1}{k-1} \zeta(2r)\zeta(k+t), \\
E &= (-1)^{i+j}\left[-S_{i,j+k} - S_{j,i+k} - \zeta(j)S_{i,k} - \zeta(i)S_{j,k}\right. \\
&\qquad \left. + \zeta(i+j+k) + \zeta(i+k)\zeta(j) + \zeta(j+k)\zeta(i) + \zeta(i)\zeta(j)\zeta(k)\right], \\
F &= \sum_{p+q+r=i+j+k} \zeta(2r)\lambda_p^{(i)}\lambda_q^{(j)},
\end{aligned}
$$

and

$$
\lambda_0^{(i)} = 1, \quad \lambda_1^{(i)} = \lambda_2^{(i)} = \cdots = \lambda_{i-1}^{(i)} = 0, \quad \lambda_{i+t}^{(i)} = (-1)^i \zeta(t) \binom{t+i-1}{i-1}.
$$

The summations are over the indices ≥ 0. *One has to replace* $\zeta(0)$ *by* $-\frac{1}{2}$, *and* $\zeta(1)$ *by* 0.

Corollary 1 (Borwein and Girgensohn). *Let* $c > 1$. *If the weight* $a+b+c$ *is even, the triple zeta function* $\zeta(a,b,c) = \sum_{0<n_1<n_2<n_3} \frac{1}{n_1^a n_2^b n_3^c}$ *can be reduced to linear Euler sums.*

Theorem 5. *(i) The cubic expression* $\sum_{n=1}^{\infty} \frac{(H_n)^3}{n^q} - 3\sum_{n\geq 1} \frac{H_n H_n^{(2)}}{n^q}$ *can be expressed in terms of the zeta values, for any weight.*

(ii) For even weights, $\sum_{n=1}^{\infty} \frac{(H_n)^3}{n^q}$ *can be computed in terms of* $S_{2,q+1}$ *and polynomials in the zeta values.*

As a consequence, this gives a proof of conjectures of Bailey, Borwein and Girgensohn:

Corollary 2. *We have*

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{(H_n)^3}{(n+1)^2} &= \frac{15}{2}\zeta(5) + \zeta(2)\zeta(3) \\
\sum_{n=1}^{\infty} \frac{(H_n)^3}{(n+1)^3} &= -\frac{33}{16}\zeta(6) + 2\zeta(3)^2 \\
\sum_{n=1}^{\infty} \frac{(H_n)^3}{(n+1)^4} &= \frac{119}{16}\zeta(7) - \frac{33}{4}\zeta(3)\zeta(4) + 2\zeta(2)\zeta(5) \\
\sum_{n=1}^{\infty} \frac{(H_n)^3}{(n+1)^6} &= \frac{197}{24}\zeta(9) - \frac{33}{4}\zeta(4)\zeta(5) - \frac{37}{8}\zeta(3)\zeta(6) + \zeta(3)^3 + 3\zeta(2)\zeta(7).
\end{aligned}
$$

3. Other relations?

If the reader wants to discover other relations, including relations on alternating Euler sums, read the details of the proofs, check that he was able to discover tricky integration contours, or know where some of these relations naturally occur in theoretical computer science, he should read this very nice paper. He will certainly enjoy it.

Bibliography

[1] Flajolet (Philippe) and Salvy (Bruno). – *Euler Sums and Contour Integral Representations.* – Research Report n° 2917, Institut National de Recherche en Informatique et en Automatique, June 1996.