Partitions of Integers: Asymptotics

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[summary by Philippe Dumas and Bruno Salvy]

Abstract

The study of the asymptotics of the number of partitions of integers under various constraints is a very rich area initiated by two papers of Hardy and Ramanujan. Some of this literature is surveyed here.

If $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\nu$ are positive integers, their sum $n = \lambda_1 + \lambda_2 + \cdots + \lambda_\nu$ is called a *partition* of n into ν summands (or parts). The number of partitions of n is denoted $p(n)$ or p_n. When there is no constraint on the λ_i, it is easy to see that the generating function of the p_n's satisfies the following identity due to Euler:

$$\mathcal{P}(q) = \sum_{n \geq 0} p_n q^n = \prod_{k>0} \frac{1}{1-q^k}. \tag{1}$$

Euler's pentagonal theorem also gives a formula for the reciprocal of this generating function:

$$\prod_{k>0}(1-q^k) = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(3m-1)/2}.$$

This last relation yields a simple way to compute the number p_n by recurrence. Numerous other relations on partitions and their congruence properties can be derived from identities on generating functions. See in particular [1].

1. Origins

The asymptotic analysis of the generating function $\mathcal{P}(q)$ is very difficult. There are singularities at all roots of unity, which implies that the circle of convergence is a natural boundary. It can be proved that a saddle-point method applies. The coefficient p_n is given by the contour integral

$$p_n = \frac{1}{2i\pi} \int_\gamma \frac{\mathcal{P}(q)}{q^{n+1}}\, dq,$$

and the main contribution comes from the neighbourhood of 1, which yields

$$p_n \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right). \tag{2}$$

Then the next contribution comes from the neighbourhood of -1, then from the neighbourhood of $\exp(\pm 2i\pi/3)$, etc. Thus the contour of integration has to go through an infinity of saddle-points, whose contribution to the integral have to be estimated. It turns out that these contributions are related by a modular transform. For, the generating function $\mathcal{P}(q)$ is related to Dedekind's η function:

$$\eta(\tau) = e^{i\pi\tau/12} \prod_{m=1}^{\infty} (1 - e^{2i\pi m\tau}) = \frac{e^{i\pi\tau/12}}{\mathcal{P}(e^{i\pi\tau})}.$$

The final result is the following theorem [9].

THEOREM 1. *The number $p(n)$ of partitions satisfies*

$$p(n) = \sum_{q=1}^{\nu} A_q \psi_q + O(n^{-1/4}),$$

where

$$\psi_q = \frac{\sqrt{q}}{2\pi\sqrt{2}} \frac{d}{dn} \left(\frac{\exp\left(\frac{\pi}{q}\sqrt{\frac{2}{3}}\sqrt{n - \frac{1}{24}}\right)}{\sqrt{n - \frac{1}{24}}} \right), \qquad A_q = \sum_{p \wedge q = 1, p \leq q} \omega_{p,q} e^{-2np i\pi/q},$$

and $\omega_{p,q}$ is a certain $24q$th root of unity.

This result is very precise: since the $O()$ term tends to 0 and the number $p(n)$ is an integer, it is sufficient to consider finitely many terms of this asymptotic expansion to compute the exact value of $p(n)$. In practice, the number of necessary terms is quite small. Theorem 1 has been refined by H. Rademacher [15] to obtain a full asymptotic expansion which is convergent. Other special types of partitions have been treated the same way. All these works rely on the theory of modular functions.

Wright followed the way opened by Hardy et Ramanujan in several works [20, 21, 22]. For instance, he studied the asymptotics of plane partitions, with generating function

$$\sum_{n \geq 0} p_{\text{plane}}(n)\, q^n = \prod_{\ell \geq 1} \frac{1}{(1 - q^{\ell})^{\ell}}.$$

The result has the following form

$$p_{\text{plane}}(n) \sim \frac{K}{n^{25/36}} \exp\left(C n^{2/3}\right),$$

which should be compared to (2) for ordinary partitions.

All these results are obtained by a saddle-point method combined with a Mellin transform.

2. Mahler's partition problem

In [12] Mahler studies the partitions whose summands are constrained to be powers of some integer $r \geq 2$. In that case, the generating function becomes

$$\prod_{k > 0} \frac{1}{1 - q^{r^k}} = \sum p_r(n) q^n = \mathcal{P}_r(q).$$

Mahler computes an expansion of $\log p_r(n)$, whose error term is a $O(1)$. This expansion shows that $p_r(n)$ is essentially of order $\exp(\log^2 n / 2 \log r)$. The basic tool is a functional equation

$$\frac{f(z + \omega) - f(z)}{\omega} = f(qz), \qquad \text{with } q = 1/r.$$

The result was improved by de Bruijn [5], using a Mellin transform approach to the logarithm, followed with a saddle-point method. Besides, in de Bruijn's work, $r > 1$ can be any real number and Mahler's error term is expressed as the sum of an oscillating series. This oscillating behaviour is studied in more detail by Erdös and Richmond in [7, 16].

3. Saddle-point method

It it quite lucky that in the case of unrestricted ordinary partitions the whole computation provides an asymptotic convergent series. If one adds constraints on the summands of the partitions it is in general not possible anymore to derive a convergent asymptotic estimate of this form. In these cases, only the saddle-point close to 1 is considered and its contribution to the integral is often itself an infinite sum.

Meinardus [1, 13] gives some general conditions which ensure that the saddle-point method works. He considers a generating function

$$\prod_{k\geq 1}\frac{1}{(1-q^k)^{a_k}},$$

where the numbers a_n are real nonnegative, and the conditions concern the Dirichlet series $D(s) = \sum_{k\geq 1} a_k/k^s$, which extends as a meromorphic function to the left of its abscissa of convergence.

Roth and Szekeres [18] study a generating function

$$\prod_{k\geq 1}(1+q^{\lambda_k}).$$

They assume the limit $s = \lim_{k\to\infty}\log\lambda_k/\log k$ exists, and use some arithmetical conditions on the summands λ_k. Their result was extended by Richmond [17], who gives several sets of conditions. As an example, Roth and Szekeres give the following expansion for the number of partitions into distinct prime summands,

$$\log q_{\text{prime}}(n) = \pi\sqrt{\frac{2}{3}}\sqrt{\frac{n}{\log n}}\left(1+O\left(\frac{\log\log n}{\log n}\right)\right).$$

The works of Meinardus and of Roth and Szekeres use the saddle-point method. The differences between them is rather a matter of style. Meinardus studies the behaviour of the generating function in the neighborhood of 1 using a Mellin transform; this gives an approximate saddle-point equation and an approximate saddle-point; next the Cauchy integral is studied. Roth and Szekeres directly use the saddle-point method and their result is expressed in an implicit manner; every application needs an auxiliary computation, in some cases with the Euler-McLaurin formula or with the Mellin transform, to obtain an explicit expansion.

4. Tauberian method

In [10], Ingham asks for a set of conditions *not highly extravagant* which leads to a result about the asymptotic behaviour of the number of partitions. He considers a sequence of real numbers $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ and its count function $\Lambda(u) = |\{\lambda_k; \lambda_k \leq u\}|$. The use of this function is natural because the generating function

$$\mathcal{P}(e^{-s}) = \prod_{k\geq 1}\frac{1}{1-e^{\lambda_k s}} = \sum_{\ell} p(\ell)e^{-\ell s}$$

and the count function are related by

$$\log\mathcal{P}(e^{-s}) = \int_0^{+\infty}\log\frac{1}{1-e^{-su}}\,d\Lambda(u).$$

Under the hypothesis

$$\Lambda(u) = Bu^\beta + R(u), \qquad \int_0^u \frac{R(v)}{v}\, dv \underset{u\to\infty}{=} b\log u + c + o(1),$$

he proves that

$$P_h(u) \sim K u^{(a-1/2)(1-\alpha)-1/2} \exp(C u^\alpha), \qquad \text{with } \alpha = \beta/(\beta+1),$$

for some explicit constants K and C. Here the function $P_h(u)$ generalises the function $p(n)$ we used previously; precisely, if $P(u)$ is the number of solutions in nonnegative integers of the inequation $n_1\lambda_1 + n_2\lambda_2 + \cdots + n_r\lambda_r + \cdots < u$ then for positive h, $P_h(u) = [P(u+h) - P(h)]/h$. Hence if $h = 1$ and the summands λ_k are integers, $P_h(n)$ is simply $p(n)$. The function $P(u)$ already appears in the work of Mahler, because it satisfies the equations $p(rm) = P(m+0)$ and $P(u) - P(u-1) = P(u/r)$ in that case.

The proof relies on a special Tauberian theorem. Indeed, the generating function appears to be a Laplace transform,

$$\mathcal{P}(e^{-s}) = \int_0^\infty e^{-su}\, dP(u).$$

The Tauberian theorem of Ingham provides an estimate of $P(u)$ in terms of $\phi(s) = \log \mathcal{P}(e^{-s})$ and the solution σ_u of the equation $\phi'(\sigma_u) + u = 0$ (which can be seen as a saddle-point equation).

The proof of Ingham works for $P(u)$ without any further condition, but for $P_h(u)$ one of the hypotheses is the monotonicity of this function. Auluck and Haselgrove [2] have extended the result of Ingham, and removed some of his hypotheses. Bateman and Erdös [3] have shown that for integer summands λ_k the function $p(n) = P_1(n)$ is monotonic if and only the set of summands has the following property: there are at least two λ's and if one removes any λ the remaining λ's have greatest common divisor unity.

5. Weak results

Hardy and Ramanujan [8] study the number $Q(x)$ of solutions of the inequation

$$2^{a_2} 3^{a_3} 5^{a_5} \cdots p^{a_p} \cdots \le x$$

into integers satisfying $a_2 \ge a_3 \ge \cdots \ge a_p \ge \cdots$. The numbers 2, 3, ..., p, ... are the prime numbers. If λ_k is the sum of the logarithms of the k first prime numbers, $Q(x)$ is essentially $P(\log x)$. They prove that

$$\log Q(x) \underset{x\to\infty}{=} \frac{2\pi}{\sqrt{3}} \sqrt{\frac{\log x}{\log\log x}} + o(1).$$

Such a result, which gives an equivalent of $\log P(u)$, is called a weak result.

The tools used by Hardy and Ramanujan is a Tauberian theorem; under some simple conditions this theorem says that

$$\log A_n \underset{n\to\infty}{=} B \ell_n^{\alpha/(1+\alpha)} / \log^{\beta/(1+\alpha)} \ell_n$$

if the behaviour near 0 of the logarithm of the Laplace transform

$$f(s) = \sum_{n\ge 1} a_n e^{-\ell_n s} = \int_0^\infty e^{-su}\, dA(s)$$

is known, namely

$$\log f(s) \underset{s\to 0}{=} \frac{A}{s^\alpha \log^\beta(1/s)}.$$

In these formulæ A_n is the summatory function

$$A_n = a_1 + a_2 + \cdots + a_n, \qquad A(x) = A_n \quad \text{for } n \le x < n+1.$$

The result is applied to the generating function

$$\mathcal{P}(e^{-s}) = \prod_{k\ge 1} \frac{1}{1 - e^{-\lambda_k s}},$$

which satisfies

$$\log \mathcal{P}(e^{-s}) \underset{s\to 0}{=} \frac{\pi^2}{6s\log(1/s)},$$

with $\ell_n = \log n$.

Brigham [4] extends the work of Hardy and Ramanujan, by considering the generating function

$$\mathcal{P}(e^{-s}) = \prod_{k\ge 1} \frac{1}{(1 - e^{-ks})^{\gamma_k}},$$

and the following hypothesis about the count function

$$\Lambda(u) = \sum_{k\le u} \gamma_k \underset{u\to\infty}{\sim} K u^\alpha \log^\beta u, \qquad \alpha > 0.$$

Two students of Bateman, Kohlbecker [11] first, and Parameswaran [14] next, consider the functional relation between the count function $\Lambda(u)$ and the summatory function $P(u)$,

$$\log \int_0^\infty e^{-su}\, dP(u) = \int_0^\infty \log \frac{1}{1 - e^{-su}}\, d\Lambda(u).$$

Kohlbecker shows the following behaviours are equivalent

$$\Lambda(u) \sim u^\alpha L(u), \qquad \log P(u) \sim u^{\alpha/(1+\alpha)} L^*(u), \qquad (\alpha > 0).$$

The function $L(u)$ and $L^*(u)$ are slowly varying, that is $L(cu) \sim L(u)$ for every $c > 0$. Moreover (L, L^*) is a dual pair; in every concrete case, $L^*(u)$ is explicitely computable from $L(u)$. The way from $P(u)$ to $\Lambda(u)$ is an Abelian theorem, and the way form $\Lambda(u)$ to $P(u)$ is a Tauberian theorem, like in the work of Hardy and Ramanujan.

Schwarz [19] gives a result which is surprising by its simplicity. The count function $\Lambda(u)$ tends to infinity (as we assumed in all preceding assertions) and satisfies $\Lambda(2u) = O(\Lambda(u))$ as $u \to \infty$. Under this hypothesis the behaviour of $\log P(u)$ is given by

$$\log P(u) \underset{u\to\infty}{=} \phi(\sigma_u) + u\sigma_u + O\left(u\sigma_u \sqrt{\psi(\sigma_u) \log \frac{1}{\psi(\sigma_u)}}\right),$$

where σ_u is the solution of the equation $\phi(\sigma) + u = 0$ for u large, and

$$\phi(\sigma) = \sum_{k\ge 1} \log \frac{1}{1 - e^{\lambda_k \sigma}}, \qquad \psi(\sigma) = \frac{\phi''(\sigma)}{|\phi'(\sigma)|^2}.$$

Schwarz gives a host of examples: ordinary partitions, $\lambda_k = k$, $\Lambda(u) \sim u$; partitions into prime numbers, $\lambda_k = p_k$, $\Lambda(u) \sim u/\log u$; partitions into rth powers, $\lambda_k = k^r$, $\Lambda(u) \sim u^{1/r}$; Mahler partitions, $\lambda_k = r^k$, $\Lambda(u) \sim \log_r u$; partitions whose summands are $\lambda_k = k^k$ or $k!$, $\Lambda(u) \sim \log u / \log\log u$, for example.

Conclusion

There is a wealth of papers on this subject. Parameters of partitions such as the number of summands can also be treated by the same kind of subject, although the computations are generally more technical. This is the problem that started Ph. Dumas in this domain, see [6] for details.

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