# Lecture Hall Partitions

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**Abstract**

A well-known theorem of Euler [2, Chap. 16] says that the number of partitions of an integer $N$ into distinct parts is equal to the number of partitions of $N$ into odd parts. The talk gives a finite version of this theorem that says that the number of "lecture hall partitions of length $n$" of $N$ equals the number of partitions of $N$ into small odd parts: $1, 3, 5, \ldots, 2n-1$. This work is a common work with Kimmo Eriksson [1].

## 1. Lecture hall partitions

Let $\mathcal{D}$ be the set of integer partitions with distinct parts. For $n \geq 1$, let $\mathcal{L}_n$ be the following set of partitions (having possibly some empty parts):

$$\mathcal{L}_n = \{(\lambda_1, \ldots, \lambda_n) : 0 \leq \lambda_1/1 \leq \lambda_2/2 \leq \cdots \leq \lambda_n/n\}.$$

We call the members of $\mathcal{L}_n$ *lecture hall partitions of length* $n$, since they describe all possible ways of designing a lecture hall with space for up to $n$ rows of seats placed on integer heights, such that at every seat there is a clear view of the speaker without obstruction from the seats in front (Figure 1).

Removing the empty parts puts $\mathcal{L}_n$ in one-to-one correspondence with the following subset of $\mathcal{D}$:

$$\mathcal{D}_n = \left\{(\mu_1, \mu_2, \ldots, \mu_m) : m \leq n \quad \text{and} \quad 0 < \frac{\mu_1}{n-m+1} \leq \frac{\mu_2}{n-m+2} \leq \cdots \leq \frac{\mu_m}{n}\right\}.$$

We will prove the following remarkable theorem.

Theorem 1 (Lecture Hall Theorem). *The generating function for lecture hall partitions of length $n$ is*

$$L_n(q) = \sum_{\lambda \in \mathcal{L}_n} q^{|\lambda|} = \prod_{i=0}^{n-1} \frac{1}{1 - q^{2i+1}},$$

*where the weight $|\lambda|$ of a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ is $\lambda_1 + \cdots + \lambda_m$.*

Equivalently, the generating function for the partitions of $\mathcal{D}_n$ is $L_n(q)$. Observe that $\mathcal{D}_n \subset \mathcal{D}_{n+1}$ and $\mathcal{D} = \lim_{n\to\infty} \mathcal{D}_n$, so in the limit this theorem yields the familiar Euler identity [2, Chap. 16]: the generating function for the elements of $\mathcal{D}$ is equal to the generating function for the elements of $\mathcal{O}$, the set of integer partitions with odd parts:

$$\sum_{\mu \in \mathcal{D}} q^{|\mu|} = \prod_{i \geq 1} (1 + q^i) = \prod_{i \geq 1} \frac{1 - q^{2i}}{1 - q^i} = \prod_{i \geq 0} \frac{1}{1 - q^{2i+1}} = \sum_{\mu \in \mathcal{O}} q^{|\mu|}.$$

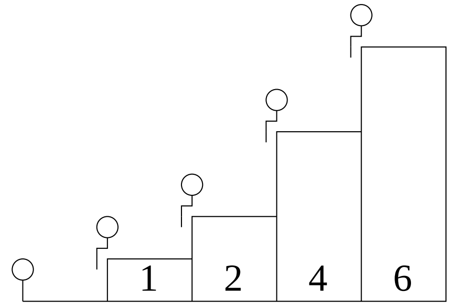


FIGURE 1. The design of a lecture hall of four rows corresponding to the lecture hall partition (1,2,4,6).

We will prove a refinement of the Lecture Hall Theorem. We define the *even* and *odd* weights $|\lambda|_e$ and $|\lambda|_o$ of a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ by

$$|\lambda|_e = \sum_{0 \le k \le \lfloor (n-1)/2 \rfloor} \lambda_{n-2k} \qquad \text{and} \qquad |\lambda|_o = \sum_{0 \le k \le \lfloor n/2 \rfloor - 1} \lambda_{n-2k-1}.$$

Of course, $|\lambda| = |\lambda|_e + |\lambda|_o$. We will prove the bivariate identity

$$\sum_{\lambda \in \mathcal{L}_n} x^{|\lambda|_e} y^{|\lambda|_o} = \prod_{i=0}^{n-1} \frac{1}{1 - x^{i+1} y^i}.$$

This identity is a corollary of Theorem 3 in section 4, taking $k = l = 2$.

We will in fact discuss a generalization to other sets of partitions of the form $\{(\lambda_1, \lambda_2, \ldots, \lambda_n) : 0 \le \lambda_1/a_1 \le \lambda_2/a_2 \le \cdots \le \lambda_n/a_n\}$ where $(a_1, a_2, \ldots, a_n)$ is a given non-decreasing sequence of integers. We define now $\mathcal{L}_n$ and $S_{(a_1, a_2, \ldots, a_n)}$ as:

$$\mathcal{L}_n = \{(\lambda_1, \ldots, \lambda_n) : 0 \le \lambda_1/a_1 \le \lambda_2/a_2 \le \cdots \le \lambda_n/a_n\} \qquad \text{and} \qquad S_{(a_1, a_2, \ldots, a_n)} = \sum_{\lambda \in \mathcal{L}_n} q^{|\lambda|}.$$

Here are surprisingly simple values of $S_{(a_1, a_2, \ldots, a_n)}$:

$$S_{1,2,5,8} = \frac{1}{(1-q)(1-q^3)(1-q^8)(1-q^{13})},$$
$$S_{1,2,5,8,19} = \frac{1}{(1-q)(1-q^4)(1-q^7)(1-q^{11})(1-q^{27})},$$
$$S_{1,2,5,8,19,30} = \frac{1}{(1-q)(1-q^3)(1-q^8)(1-q^{13})(1-q^{31})(1-q^{49})},$$
$$S_{1,2,7,12,41} = \frac{1}{(1-q)(1-q^5)(1-q^9)(1-q^{31})(1-q^{53})}.$$

## 2. Reduction of lecture hall partitions

Fix a non-decreasing sequence $a = (a_i)_{i \ge 1}$ of positive integers, and fix a positive integer $n$. An $n$-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{N}^n$ is a lecture Hall partition if and only if $\lambda_i \ge \lceil \lambda_{i-1} a_i / a_{i-1} \rceil$ for $2 \le i \le n$. For $1 \le i \le n$, let $\lambda^{(i)} = (0, \ldots, 0 a_i, a_{i+1}, \ldots, a_n) \in \mathbb{N}^n$. If $\lambda$ belongs to $\mathcal{L}_n$, then the sum $\lambda + \lambda^{(i)}$ also belongs to $\mathcal{L}_n$.

LEMMA 1. *Let $\lambda$ be a lecture hall partition belonging to $\mathcal{L}_n$. Then $\lambda - \lambda^{(i)}$ belongs to $\mathcal{L}_n$ if and only if $\lambda_i - \lceil \lambda_{i-1} a_i / a_{i-1} \rceil \ge a_i$ for $1 \le i \le n$.*

Definition 1. A lecture hall partition of length $n$ is said to be *reduced* if $0 \leq \lambda_i - \lceil \lambda_{i-1} a_i / a_{i-1} \rceil < a_i$ for $1 \leq i \leq n$. The set of reduced partitions of $\mathcal{L}_n$ will be denoted by $\mathcal{R}_n$.

Lemma 2. *Let $\lambda$ be a lecture hall partition of length $n$. Then there exists a unique reduced lecture hall partition $\mu$ and a unique sequence of integers $(k_i)_{1 \leq i \leq n}$ such that $\lambda = \mu + \sum_{i=1}^{n} k_i \lambda^{(i)}$.*

Consequently, the generating function for lecture hall partitions of length $n$ is

$$S_n = \sum_{\lambda \in \mathcal{L}_n} x^{|\lambda|_e} y^{|\lambda|_o} = \frac{P_n(x,y)}{\prod_{i=1}^{n} (1 - x^{|\lambda^{(i)}|_e} y^{|\lambda^{(i)}|_o})}$$

where the polynomial $P_n(x,y) = \sum_{\mu \in \mathcal{R}_n} x^{|\mu|_e} y^{|\mu|_o}$ enumerates reduced lecture hall partitions.

## 3. An involution on $\mathcal{R}_n$

For $\mu \in \mathcal{R}_n$, let $\mu^* = (\mu_1^*, \ldots, \mu_n^*)$ be the unique $n$-tuple such that

$$\begin{cases} \mu^*_{n-2k} = \mu_{n-2k} & \text{for } n - 2k \geq 1 \\ \mu^*_{n-2k-1} - \left\lceil \frac{a_{n-2k-1}}{a_{n-2k-2}} \mu^*_{n-2k-2} \right\rceil = \left\lfloor \frac{a_{n-2k-1}}{a_{n-2k}} \mu_{n-2k} \right\rfloor - \mu_{n-2k-1} & \text{for } n - 2k - 1 \geq 1. \end{cases}$$

Theorem 2. *The correspondence $\mu \mapsto \mu^*$ defines an involution on the set $\mathcal{R}_n$.*

We can extend the involution $\mu \mapsto \mu^*$ into a bijection $f$ from $\mathcal{R}_n \times [0, a_{n+1}]$ onto $\mathcal{R}_{n+1}$, by defining

$$f(\mu_1, \ldots, \mu_n; i) = \left( \mu_1^*, \ldots, \mu_n^*, \left\lceil \frac{a_{n+1}}{a_n} \mu_n^* \right\rceil + i \right).$$

It is clear that:

$$\begin{aligned} |f(\mu,i)|_o &= |\mu^*|_e = |\mu|_e \,, \\ |f(\mu,i)|_e &= i - |\mu|_o + \sum_k \left( \left\lceil \frac{a_{n-2k+1}}{a_{n-2k}} \mu_{n-2k} \right\rceil + \left\lfloor \frac{a_{n-2k-1}}{a_{n-2k}} \mu_{n-2k} \right\rfloor \right). \end{aligned}$$

## 4. The $(k-l)$-sequences

By a $(k-l)$-sequence we shall mean a sequence $a$ defined by the initial values $a_1 = 1$ and $a_2 = l$ and the following recurrence relations:

$$\begin{cases} a_{2n} = l a_{2n-1} - a_{2n-2} & \text{for } n \geq 2 \\ a_{2n+1} = k a_{2n} - a_{2n-1} & \text{for } n \geq 1 \end{cases}$$

where $k, l \geq 2$ are two integers. We obtain

$$\begin{aligned} |f(\mu,i)|_o &= |\mu|_e \,, \\ |f(\mu,i)|_e &= i - |\mu|_o + \begin{cases} k|\mu|_e & \text{if } n \text{ is even,} \\ l|\mu|_e & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

This implies that the generating functions $P_n(x,y) = \sum_{\mu \in \mathcal{R}_n} x^{|\mu|_e} y^{|\mu|_o}$ for reduced lecture hall partitions can be computed inductively via the following recurrence relations:

$$P_{2n+1}(x,y) = \frac{1 - x^{a_{2n+1}}}{1 - x} P_{2n}(x^k y, x^{-1}) \qquad \text{and} \qquad P_{2n}(x,y) = \frac{1 - x^{a_{2n}}}{1 - x} P_{2n-1}(x^l y, x^{-1})$$

with the initial condition $P_0 = 1$.

The sequence $a^*$ is defined by $a^*_0 = 0$, $a^*_1 = 1$ and the recurrence relations:

$$\begin{cases} a_{2n} = l a_{2n-1} - a_{2n-2} & \text{for } n \geq 2 \\ a_{2n+1} = k a_{2n} - a_{2n-1} & \text{for } n \geq 1. \end{cases}$$

THEOREM 3. *Given a $(k,l)$-sequence $a$, the generating functions $S_n = \sum_{\lambda \in \mathcal{L}_n} x^{|\lambda|_e} y^{|\lambda|_o}$ for lecture hall partitions of even and odd length are given by:*

$$S_{2n} = \prod_{i=1}^{2n} \frac{1}{1 - x^{a_i} y^{a^*_i}} \qquad \text{and} \qquad S_{2n+1} = \prod_{i=1}^{2n+1} \frac{1}{1 - x^{a^*_{i+1}} y^{a_{i-1}}}.$$

## 5. Limit theorems

Taking the limit $n \to \infty$ in Theorem 3 leads to the following results:

THEOREM 4. *For $k \in \mathbb{N}$ and $k \geq 2$, the bivariate generating function of partitions $(\mu_1, \ldots, \mu_n)$ such that $\frac{\mu_{i+1}}{\mu_i} > \frac{k + \sqrt{k^2-4}}{2}$ is:*

$$\sum_{\mu} x^{|\mu|_e} y^{|\mu|_o} = \prod_{i \geq 1} \frac{1}{1 - x^{a_i} y^{a_{i-1}}}$$

*with $a_0 = 0$, $a_1 = 1$ and $a_{i+1} = k a_i - a_{i-1}$.*

THEOREM 5. *For $k \in \mathbb{N}$ and $k \geq 2$, the generating function of partitions $(\mu_1, \ldots, \mu_n)$ such that $\frac{\mu_{i+1}}{\mu_i} > \frac{k + \sqrt{k^2-4}}{2}$ is:*

$$\sum_{\mu} q^{|\mu|} = \prod_{i \geq 1} \frac{1}{1 - q^{e_i}}$$

*with $e_1 = 1$, $e_2 = k+1$ and $e_{i+1} = k e_i - e_{i-1}$.*

EXAMPLE. $k = 2$. In that case $\mu_{i+1} > \mu_i$ and we obtain the Euler identity [2, Chap. 16]:

$$\sum_{\mu \in \mathcal{D}} q^{|\mu|} = \prod_{i \geq 0} \frac{1}{1 - q^{2i+1}}.$$

EXAMPLE. $k = 3$. In that case $\mu_{i+1} > \frac{3+\sqrt{5}}{2} \mu_i$ and:

$$\sum_{\mu \in \mathcal{L}_n} q^{|\mu|} = \frac{1}{(1-q)(1-q^4)(1-q^{11})(1-q^{29})(1-q^{76}) \cdots} = \prod_{i \geq 1} \frac{1}{(1 - q^{e_i})},$$

with $e_1 = 1$, $e_2 = 4$ and $e_{i+1} = 3e_i - e_{i-1}$. In fact $e_i = F_{2i-3} + F_{2i-1}$ where $F_i$ is the $i$th Fibonnaci number.

## 6. Questions

(1) Give a characterization of the sequences $(a_1, \ldots, a_n)$ that have a simple expression for the corresponding generating functions.
(2) Find finite version of other theorems like the Rogers-Ramanujan theorem for instance.

## Bibliography


[1] Bousquet-Mélou (M.) and Eriksson (K.). – *Lecture Hall Partitions.* – Rapport LaBRI n° 1107-96, Laboratoire Bordelais de Recherche en Informatique, Université de Bordeaux I, 1996.
[2] Euler (L.). – *Introductio in analysin infinitorum.* – Marcum-Michælem Bousquet, Lausannæ, 1748.