Symbolic and Numerical Manipulations of Divergent Power Series

Jean Thomann
IN2P3, Strasbourg

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[summary by Bruno Salvy]

Abstract

Divergent series arise naturally in many different contexts. This talk describes mixed symbolic-numerical algorithms to deal with these series when they arise from linear differential equations.

Introduction

A simple example of a divergent numerical series is obtained when summing a Taylor series outside its circle of convergence. More violent divergence is encountered when solving a linear differential equation like the Euler equation

$$x^2y' + y = x$$

by an undeterminate coefficient method. The power series one obtains is the Euler series

$$\sum_{n\geq 0}(-1)^n n!x^{n+1},$$

which has a radius of convergence equal to 0. This problem also occurs in non-linear differential equations, singular perturbations, difference equations, or asymptotic analysis (e.g., by the Laplace method).

The Borel-Ritt theorem states that any power series on any sector of finite opening in the complex plane is the asymptotic expansion of a function which is analytic in the sector. However, this analytic function is far from being uniquely determined, which makes numerical evaluation hopeless. In the context of differential equations the situation is much better because of the following result.

Theorem 1. *Let $G(x, y_0, \ldots, y_n)$ be an analytic function of $n + 2$ variables and $\hat{f} \in \mathbb{C}[[x]]$ a formal power series solution of $G(x, y, \ldots, y^{(n)}) = 0$. Then there exists a real number $k > 0$ such that for all open sectors V with vertex at the origin, opening $< \pi/k$ and small enough radius, there exists a function f which is a solution of the differential equation $G(x, y, \ldots, y^{(n)}) = 0$ asymptotic to $\hat{f}$ on V.*

Thus the main numerical problem is to devise techniques that will sum the divergent series not to values of any analytic function asymptotic to it, but to values of the actual solution of the differential equation corresponding to it.

1. Elementary methods

To compute the sum of a convergent power series $\sum_n a_n x^n$ outside its circle of convergence, one first has to define a path connecting the origin to the point where the sum is desired. A basic subproblem is that of summing along a ray originating at the origin and avoiding singularities of the function. Lindelöf gave a simple way of doing this by computing this value as the limit of

$$a_0 + \lim_{t\to 0} \sum_{n\geq 1} a_n x^n e^{-tn\log n}.$$

When the series is convergent at x, the result is the sum of the series. This technique was generalized by Hardy to sum divergent series of the same type as the Euler series, by computing

$$a_0 + a_1 x + a_2 x^2 + \lim_{t\to 0} \sum_{n\geq 3} a_n x^n e^{-t\log n \log\log n}. \tag{1}$$

Unfortunately, this technique does not behave very well numerically.

The simplest efficient technique to deal with divergent series of the type of the Euler series is called *summation to the least term*. For instance, the values of the successive terms of the Euler series at $x = 1/10$ are

$$.100000, -.010000, .002000, -.000600, .000240, -.000120, .000072, -.000050, .000040, -.000036,$$
$$.000036, -.000040, .000048, -.000062, .000087, -.000131, .000209, -.000356, .000640, -.001216.$$

The absolute value of the terms first decrease, then reaches a minimum at 0.000036, and eventually increase to infinity. By summing these terms up to the smallest one, one gets the numerical value 0.09154563200, which is very close to the value of the corresponding *function* solution of the differential equation, namely 0.09156333394. Using a convergent integral representation for the Euler series, it is not difficult to show (see [5]) that the error made by truncating this series at its least term is exponentially small (with respect to $1/x$). This property is actually much more general (see below). The drawback of this good precision is the impossibility of obtaining an arbitrary precision by this method. This is to be contrasted with the direct summation of convergent power series, where the terms generally first increase before decreasing to 0, but numerous terms are necessary to obtain a good precision. As a consequence, many techniques to convert from various representations of a function to a divergent series have been developed [3].

2. Gevrey asymptotics, Borel transform, k-summability

A good framework to account for the nice behaviour of common divergent series is provided by *Gevrey asymptotics*. A *Gevrey series* is a power series whose coefficients' growth is bounded by $C(n!)^{1/k}A^n$, for some fixed $C, A, k > 0$. Gevrey asymptotic expansions are Gevrey series for which the remainder term satisfies the same type of bound. More precisely, we have the following.

DEFINITION 1. Let k be a positive real number and let V be an open sector with vertex 0. Let f be an analytic function on V. The formal power series $\hat{f} = \sum_{n\geq 0} a_n x^n$ is *Gevrey asymptotic to f of order $s = 1/k$ on V* if for all compact sub-sectors W of V and for all $n \in \mathbb{N}$, there exist $C_W > 0$ and $A_W > 0$ such that

$$|x|^{-n} \left| f(x) - \sum_{p=0}^{n-1} a_p x^p \right| < C_W (n!)^{1/k} A_W^n, \qquad \forall x \in W, \quad x \neq 0$$

By Stirling's formula, truncating a Gevrey asymptotic expansion of order $1/k$ to the least term gives an exponentially small error (in $1/x^k$). This is one of the facets of the interest of Gevrey asymptotics. Another crucial property, due to Watson, is that there is at most one analytic function f Gevrey asymptotic of order $1/k$ to a series $\hat{f}$ on a sector of opening larger than π/k. This provides the uniqueness necessary for numerical computations based on the series alone.

With the same hypotheses as in Theorem 1, an old theorem of Maillet states that there exists $k > 0$ such that $\hat{f}$ is a Gevrey series of order $1/k$. This result is useful in conjunction with a counterpart of Theorem 1 due to Ramis and Sibuya which states that if $\hat{f}$ is Gevrey of order $1/k$ then there exists $k' \geq k$ such that for any sector V with vertex 0, opening $< \pi/k'$ and sufficiently small radius, there exists a function f solution to the differential equation which is Gevrey asymptotic of order k to $\hat{f}$. Combining these two results explains why summing to the least term is a good method for formal series solutions of differential equations.

If $\hat{f} = \sum a_n x^n$ is a Gevrey series of order $1/k$, then its *Borel transform of index k* is defined as

$$(\hat{\mathcal{B}}_k \hat{f})(\xi) = \sum_{n \geq 0} \frac{a_n}{\Gamma(1 + n/k)} \xi^n.$$

For $k = 1$, this corresponds to dividing the n-th coefficient by $n!$. Estimates on the coefficients show that this transform is an analytic function $\phi(\xi)$. Then if the Laplace transform of index k

$$f(x) = \int_d k\phi(\xi) e^{-\xi^k/x^k} \xi^{k-1} d\xi$$

converges, it is called the sum of $\hat{f}$ in the direction d, where d is a straight line from 0 to infinity. The series $\hat{f}$ is then said to be *k-summable in the direction d*. The convergence of this integral is related to the growth of ϕ at infinity. It is easy to see that the Taylor series of f is precisely $\hat{f}$ so that this process yields a convergent representation for $\hat{f}$. The sum however depends on the path of integration d, in the same way an analytic continuation depends on a path. This dependency is related to the *Stokes phenomenon.*

Numerically, in the case of convergence, the problem is reduced to finding k and computing the analytic continuation of ϕ. In the case of solutions of *linear* differential equations, this computation is simplified by noticing that k can be deduced from the slopes of a Newton polygon associated with the linear differential equation and that ϕ satisfies a linear differential equation derived formally from that satisfied by $\hat{f}$. Therefore its Taylor coefficients satisfy a computable linear recurrence which can be used to obtain many coefficients efficiently. Besides, the possible singularities of ϕ are located at the zeroes of the leading coefficient of the linear differential equation it satisfies, so that it is possible to compute the continuation along a path which avoids singularities, with a knowledge of the exact radius of convergence of the power series one is computing. This process can also be applied to the divergent series that occur as part of the asymptotic expansion of solutions of linear differential equations at an irregular singular point, by first computing a linear differential equation satisfied by these series.

3. Multisummability

Not all solutions of linear differential equations are k-summable for some k. One reason for this is that the order of growth of an analytic function at infinity is related to the growth of its Taylor coefficients at the origin. Thus by adding a 1-summable and a 2-summable divergent series, one obtains a series which is Gevrey of order 1, but the growth at infinity of its Borel transform of level 1 is exponential of order 2. This leads to the consideration of a more general class of divergent series.

DEFINITION 2. Let $k_1, \dots, k_r$ be real numbers such that $k_1 > \cdots > k_r > 0$ and let d be a line from 0 to infinity. A formal power series $\hat{f}(x)$ is $(k_1, \dots, k_r)$-*summable in the direction* d if there exists a positive integer m such that $\hat{f}(x^{1/m})$ is a sum of r series $\hat{f}_1, \dots, \hat{f}_r$, each $\hat{f}_i$ being k_i/m summable in the direction d.

A result of Jurkat is that Hardy's summation technique (1) will sum any multisummable series without having to know $k_1, \dots, k_r$.

The following recent theorem due to Braaksma demonstrates the relevance of multisummability.

THEOREM 2. *Let* $G(x, y_0, \dots, y_n)$ *be an analytic function of* $n+2$ *variables and* $\hat{f} \in \mathbb{C}[[x]]$ *a formal power series solution of* $G(x, y, \dots, y^{(n)}) = 0$. *Let* $k_1 > \cdots > k_r > 0$ *be the positive slopes of the associated Newton polygon. Then* $\hat{f}$ *is* $(k_1, \dots, k_r)$ *summable in every direction d, except possibly a finite number of them.*

Braaksma's proof uses Écalle's theory of accelero-summability.

In the linear case, a technique due to Balser makes it possible to compute the sum by computing successive Borel transforms of indices κ_j related to the k_i's by $1/k_j = 1/\kappa_1 + \cdots + 1/\kappa_j$ and then recovering the function by computing the corresponding Laplace transforms of order κ_j in reverse order. At each step, exact linear differential equations can be computed for the various Taylor series and exact linear recurrences for their coefficients.

Conclusion

Numerically, the difficulty is that each level of integration is time consuming and induces a precision loss. At the moment, this process is still largely interactive, notably the choice of paths of integration at each step.

Bibliography

[1] Balser (Werner). – *From Divergent Power Series to Analytic Functions.* – Springer-Verlag, 1994, *Lecture Notes in Mathematics*, vol. 1582.

[2] Braaksma (B. L. J.). – Multisummability of formal power series solutions of nonlinear meromorphic differential equations. *Annales de l'Institut Fourier*, vol. 42, n° 3, 1992, pp. 517–540.

[3] Dingle (Robert B.). – *Asymptotic Expansions: Their Derivation and Interpretation.* – Academic Press, London, New York, 1973.

[4] Hardy (G. H.). – *Divergent Series.* – Oxford University Press, 1947.

[5] Olver (F. W. J.). – *Asymptotics and Special Functions.* – Academic Press, 1974.

[6] Ramis (J.-P.) and Sibuya (Y.). – Hukuhara domains and fundamental existence and uniqueness theorems for asymptotic solutions of Gevrey type. *Asymptotic Analysis*, vol. 2, 1989, pp. 39–94.

[7] Ramis (J.-P.) and Sibuya (Y.). – A new proof of multisummatility of formal solutions of non linear meromorphic differential equations. *Annales de l'Institut Fourier*, vol. 44, n° 3, 1994, pp. 811–848.

[8] Ramis (Jean-Pierre). – *Séries divergentes et théories asymptotiques.* – Société Mathématique de France, 1993, *Panoramas et Synthèses*, vol. 121.

[9] Thomann (Jean). – Resommation des séries formelles. Solutions d'équations différentielles linéaires du second ordre dans le champ complexe au voisinage de singularités irrégulières. *Numerische Mathematik*, vol. 58, n° 5, 1990, pp. 503–535.

[10] Thomann (Jean). – Procédés formels et numériques de sommation de séries solutions d'équations différentielles. – Preprint, April 1994.