The Solution to a Conjecture of Hardy

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[summary by Joris van der Hoeven]

Abstract

John Shackell proves a conjecture of Hardy, which states that the inverse function of $\log\log x \log\log\log x$ is not asymptotic to any exp-log function. In order to prove this, he uses his technique of nested forms.

1. Introduction

Hardy was the first to study systematically the notion of exp-log functions in the context of asymptotic expansions [3, 4]. These functions are built up from $\mathbb{Q}$ or $\mathbb{R}$ by the use of field operations, exponentiation and logarithm. Examples are

$$\exp(x + \log(x^2 + e^x)), \quad \text{and} \quad \log(\log(x^3 + \exp(1995x)) + 2).$$

He established that the sign of any exp-log function is constant in a neighbourhood of infinity. This property makes exp-log functions extremely useful for doing asymptotics. Although many functions one encounters in practice are asymptotic to some exp-log function, Hardy conjectured that this is not the case for the inverse function $\Phi(x)$ of $\log_2 x \log_3 x$, where the index denotes iteration. In other words, Φ is defined by

$$\log\log\Phi(x)\log\log\log\Phi(x) = x.$$

Now it is known since Liouville [5] that the inverse function $\Psi(x)$ of $x\log x$ is not an exp-log function. In his talk Shackell shows how to deduce Hardy's conjecture from this result.

In order to do this, Shackell uses his technique of nested expansions, which was originally designed to construct algorithms for doing asymptotics. Although Shackell also spoke about these issues in his talk, we will only recall the material which is necessary in order to prove Hardy's conjecture. For more details about the algorithmic aspects of asymptotics, we refer to [1, 2, 6, 7, 8, 9, 10, 11, 12].

2. On nested expansions

We start with some definitions. Let f_1 and f_2 be exp-log functions which tend to zero. We say that f_1 and f_2 are *comparable* or of the same asymptotic scale, if there exist positive integers m and n with $f_1 \leq f_2^m$ and $f_2 \leq f_1^n$ (recall that the germs of exp-log functions at infinity form a totally ordered field). The comparability relation is an equivalence relation and we denote the equivalence class of f by $\gamma(f)$. The equivalence classes can be ordered by $\gamma(f_1) > \gamma(f_2)$, if $f_1 \leq f_2^n$ for all positive integers n.

Let us also introduce the concept of z-functions. Such a function is one of the following:

$$\begin{aligned} \operatorname{zexp}_n(t) &= t^{-n}(\exp(t) - 1 - t - \cdots - t^n/n!), \\ \operatorname{zlog}_n(t) &= t^{-n}(\log(1+t) - t - \cdots - (-1)^{n-1}t^n/n), \\ \operatorname{zinv}_n(t) &= t^{-n}(1/(1+t) - 1 + t - \cdots - (-1)^{n-1}t^n), \end{aligned}$$

for any integer $n \geq 0$. If $t_1, \ldots, t_m$ are exp-log functions which tend to zero, we denote by $Z(t_1, \ldots, t_m)$ the set of functions which can be obtained from $t_1, \ldots, t_m$ by using addition, subtraction, multiplication and application of z-functions. Shackell proved the following theorem [8].

THEOREM 1. *Let f be an exp-log function which tends to infinity. Then there exist exp-log functions $t_1, \ldots, t_m$ with $\gamma(t_1) > \cdots > \gamma(t_m)$, such that f can be expressed as $f = \exp_r((k+z)L)$, where $\exp_r$ is the r-th iterated exponential, k is a non zero constant, L is a product of real powers of iterated logarithms, and z belongs to $Z(t_1, \ldots, t_m)$.*

The expression $f = \exp_r((k+z)L)$ is called a nested form of f. More generally, one can recursively compute nested forms for $t_1, \ldots, t_m$. Doing this, one obtains so called nested expansions. Shackell and Salvy have shown how to obtain automatically nested expansions of the functional inverses of exp-log functions [7], modulo suitable hypotheses on exp-log constants.

3. The solution to Hardy's conjecture

Denote by Ψ the inverse function of $x \log x$, and recall that $\Phi = \exp_2 \Psi$ denotes the inverse function of $\log_2 x \log_3 x$. The following lemma is crucial for the proof of Hardy's conjecture.

LEMMA 1. *There is no exp-log function f such that*

$$|f - \Psi| \leq e^{-\delta\sqrt{x}},$$

for all $\delta > 0$.

PROOF. Assume that such a function f exists. It can be shown (using the same notations as in the above theorem), that one can find $z \in Z(t_1, \ldots, t_m)$ such that

$$f = \frac{x}{\log x}(1+z).$$

Now replace all terms in the Laurent series expansion of z in $t_1, \ldots, t_m$, which have equivalence class superior or equal to $\gamma(e^{\sqrt{x}})$ by zero. Let $\hat{z}$ be the series so obtained and denote $\hat{f} = (x/\log x)(1+\hat{z})$. Then it can be shown that $\hat{f}$ is an exp-log function, so that modulo changing δ, we may assume without loss of generality that $\gamma(t_1) < \cdots < \gamma(t_m) < \gamma(e^{\sqrt{x}})$.

Now it is easily seen that

$$|f \log f - x| = |f \log f - \Psi \log \Psi| \leq e^{-\delta' \sqrt{x}},$$

for some suitable δ'. But $f \log f$ and x are both analytic functions in $x, \log x, \log_2 x, t_1, \ldots, t_m$, so that we must have $f \log f = x$. But this is impossible by Liouville's theorem. Hence, we obtained the desired contradiction. □

THEOREM 2. *There does not exist any exp-log function which is asymptotic to the inverse function of $\log_2 x \log_3 x$.*

PROOF. Since $\Psi = x/\log\Psi = x/(\log x - \log_2 \Psi)$, we have

$$\Phi = \exp_2(x/(\log x - \log_4 \Phi)).$$

Now let g be asymptotic to Φ, so that $\log g - \log \Phi = o(1)$. Then

$$\log g/\log\Phi = 1 + o(\log^{-1}\Phi) = 1 + o(\exp((\varepsilon - 1)x/\log x)),$$

for any $\varepsilon > 0$. Hence

$$|\log_2 g - \log_2 \Phi| < \exp^{-\delta\sqrt{x}},$$

for all $\delta > 0$. By the lemma, it follows that $\log_2 g$ cannot be an exp-log function. Hence neither is g. □

This theorem shows that the scale of all exp-log functions is not sufficient to do asymptotic expansions of functional inverses. This shows that one essentially needs more general asymptotic scales, or an alternative way to represent asymptotic series. One of the candidates for such an alternative way of representing series is Shackell's technique of nested expansions.

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