Combinatorial Interpretations of Continued Fractions

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[summary by Philippe Flajolet]

Abstract

This seminar describes historical motivations and a combinatorial setting for continued fraction expansions of formal power series. By general theorems, universal continued fractions are generating functions of lattice paths in the plane. This can be used either to solve counting problems in terms of continued fractions or to develop a combinatorial approach to continued fraction identities. Roblet's presentation develops this theory applied not only to standard continued fractions (the so-called J- and S-fractions), but also to Padé approximants, T-fractions, and 2-point approximants.

1. Euler

The story begins with an initial frustration of Euler —*"What to do with a divergent series of a differential equation?"*— turned into a brilliant intuition —*"Expand into continued fractions!"*. Euler's intuition was to be later brought to fruition by Stieltjes; it is historically the starting point of orthogonal polynomials, a domain that has since blossomed with rich applications to approximation theory. The talk concerns provides a historical background and then proceeds to discuss the formal aspects of numerous identities relating power series, continued fractions, orthogonal polynomials and classical combinatorial structures.

Euler starts with the purely divergent series

$$Y(z) = \sum_{n=0}^{\infty} n! z^n. \tag{1}$$

One reason for investigating this series is that it directly arises when one attempts to solve the differential equation

$$z\frac{d}{dz}(zy(z)) - y(z) + 1 = 0, \tag{2}$$

by indeterminate coefficients. What if Nature was confronting us with such an equation? Which "information" is concealed in a series like $Y(z)$?

Euler's idea is to try expanding (1) into a *continued fraction*, and he then discovers a remarkable

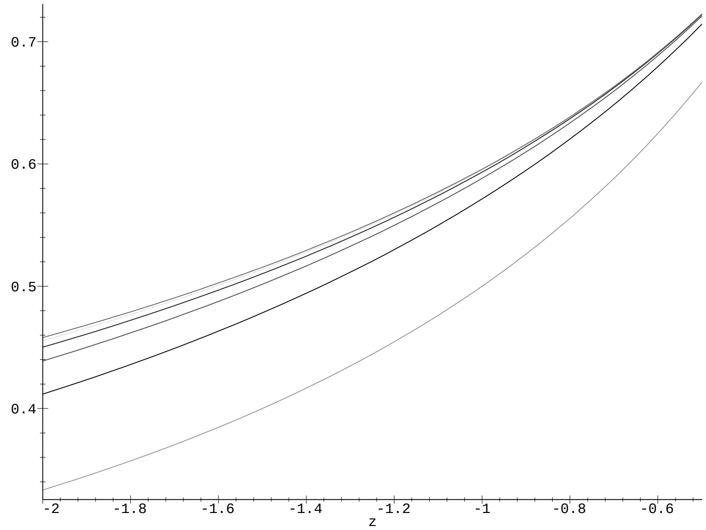

FIGURE 1. The first five convergents of Euler's continued fraction, starting with $P_1/Q_1 = 1/(1-z)$, converge quickly to a well-defined limit.

pattern,

$$\sum_{n=0}^{\infty} n!z^n = \cfrac{1}{1 - 1\cdot z - \cfrac{1^2\cdot z^2}{1 - 3\cdot z - \cfrac{2^2\cdot z^2}{1 - 5\cdot z - \cfrac{3^2\cdot z^2}{\ddots}}}}, \tag{3}$$

involving simply odd numbers and squares.

The format of (3) may seem strange at first sight but it is not. Given a formal power series $f(z) = \sum_{n=0}^{\infty} f_n z^n$, define its integral and fractional parts by

$$\lfloor f \rfloor = f_0 + f_1 z, \qquad \{f\} = f_2 + f_3 z^2 + \cdots,$$

so that

$$f = \lfloor f \rfloor + z^2\{f\}.$$

The continued fraction (3) is then obtained by applying a simple variant of the usual continued fraction algorithm to Y, very much like in the well-known arithmetic case (take inverses and iterate on fractional parts).

Returning to the original problem, one may wonder what to do with (3) in the context of standard analytic solutions to the original equation (2). One naturally looks at the sequence of *convergents*,

$$\frac{0}{1}, \quad \frac{1}{1-z}, \quad \frac{1-3z}{1-4z+2z^2}, \quad \cdots \quad \frac{P_h(z)}{Q_h(z)}, \cdots$$

Consider them for instance for $z \in [-2, -\frac{1}{2}]$. The miracle is that they do appear to converge to a well defined function, $Y_1(z)$, as shown in Figure 1.

It turns out that $Y_1(z) = \lim_{h\to+\infty} \frac{P_h(z)}{Q_h(z)}$ is an actual *analytic* solution to (2) in the complex plane slit along $[0, +\infty]$. This fact is not too hard to establish in our particular case since everything is fairly explicit: the P and Q polynomials satisfy, like any polynomials arising from convergents, a linear recurrence of order 2; as the coefficients are simple, the recurrence is solvable and we just obtain the (reciprocal polynomials) of the Laguerre polynomials and their associates.

About a century later, Stieltjes was to prove that this is a general phenomenon and he gave general conditions under which a (possibly divergent) series is numerically approximated by the particular *rational approximants*, P_h/Q_h. (At this point the connection with differential equations is lost as one addresses a much more general problem, namely summation of divergent series.)

In addition, as is well-known, the Laguerre polynomials are orthogonal with respect to the scalar product

$$\langle u, v\rangle = \int_0^\infty u(x)v(x)e^{-x}\,dx,$$

with moments

$$n! = \int_0^\infty x^n e^{-x}\,dx.$$

Hence we have the representation of the original series as a *Stieltjes transform* of e^{-x}:

$$Y_2(z) = \int_0^\infty \frac{1}{1-zx}e^{-x}\,dx.$$

Here, the well defined $Y_2(z)$ is "consistent" with $Y_1(z)$ in the sense that $Y_1(z) = Y_2(z)$ for z in the slit plane.

Once more, this is a general fact: the reciprocals of the denominator polynomials Q_h are always orthogonal (at least formally, and under Stieltjes' conditions analytically as well) with respect to a scalar product

$$\langle u, v\rangle = \int_0^\infty u(x)v(x)\,d\mu(x),$$

as shown by a simple algebraic computation. The full continued fraction is then the Stieltjes transform of the orthogonality measure $d\mu(x)$,

$$\int \frac{1}{1-zx}\,d\mu(x),$$

which may be viewed as a continuous version of a partial fraction decomposition of the original function (or its associated continued fraction).

Though orthogonal polynomials are also useful for enumerations, the rest of this presentation concentrates on enumerative uses of continued fractions. At any rate, continued fractions are the historical source of the theory of orthogonal polynomials, starting from Euler's example.

Note. Classical bibliographical sources for the theory alluded to here are the books by Perron [8] and Wall [14]. Chihara's book [1] is a useful introduction to orthogonal polynomials. Stieltjes' Collected Papers [11] have been recently reprinted with an insightful summary by Van Assche of Stieltjes' contributions.

2. Jacobi and Stieltjes

Euler's idea generalizes. Given a series

$$J(z) = 1 + \sum_{n=1}^{\infty} j_n z^n,$$

the continued fraction algorithm almost surely succeeds and delivers a continued fraction expansion of the form

$$J(z) = \cfrac{1}{1 - \kappa_0 \cdot z - \cfrac{\lambda_1 \cdot z^2}{1 - \kappa_1 \cdot z - \cfrac{\lambda_2 \cdot z^2}{1 - \kappa_2 \cdot z - \cfrac{\lambda_3 \cdot z^2}{\ddots}}}}, \tag{4}$$

called a J-fraction (for Jacobi). A variant of the J-fraction is the S-fraction (for Stieltjes)

$$S(z) = \cfrac{1}{1 - \cfrac{\lambda_1 \cdot z}{1 - \cfrac{\lambda_2 \cdot z}{1 - \cfrac{\lambda_3 \cdot z}{\ddots}}}}. \tag{5}$$

The two are closely related and it is easily recognized that

$$\text{S-fraction}[f(z)] \underset{z^2 \mapsto z}{=} \text{J-fraction}[f(z^2)].$$

Again, this notion is general as a series almost surely admits an S-fraction expansion. (Only rational functions and functions whose Taylor expansion "resembles" a rational function do not admit of such continued fraction expansions.)

3. Continued fractions and special functions

Stieltjes also determined the continued fraction for the ordinary generating function of the Bell numbers $\{B_n\}$ in the form

$$\sum_{n=0}^{\infty} B_n z^n = \cfrac{1}{1 - 1 \cdot z - \cfrac{1 \cdot z^2}{1 - 3 \cdot z - \cfrac{2 \cdot z^2}{1 - 5 \cdot z - \cfrac{3 \cdot z^2}{\ddots}}}}, \tag{6}$$

while a continued fraction expansion of Gauß related to hypergeometric functions implies

$$\sum_{n=0}^{\infty} (2n-1)!! z^n = \cfrac{1}{1 - \cfrac{1 \cdot z^2}{1 - \cfrac{2 \cdot z^2}{1 - \cfrac{3 \cdot z^2}{\ddots}}}}, \tag{7}$$

where $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$.

At this stage, it is of interest to note that (3), (6), (7) are Laplace transforms of certain exponential generating functions that all have a simple form, namely

$$\frac{1}{1-z}, \quad e^{e^z - 1}, \quad e^{z^2/2}.$$

In effect the simplest proof of these continued fractions is by means of a theorem of Stieltjes and Rogers that relates an addition formula for an exponential generating function $\phi(z)$ to the continued fraction expansion of the corresponding ordinary generating function $f(z)$. Consider therefore a Laplace pair,

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \quad \text{and} \quad \phi(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!}.$$

The Stieltjes-Rogers theorem asserts that any addition formula for ϕ in the form

$$\phi(x+y) = \sum_{k=0}^{\infty} \phi_k(x)\phi_k(y) \quad \text{where} \quad \phi_k(x) = O(x^k)$$

is simply converted into a J-fraction expansion of $f(z)$. For instance for $\phi(z) = \sec(z) = 1/\cos(z)$, the addition formula reads

$$\phi(x+y) = \frac{1}{\cos(x+y)} = \sum_{k=0}^{\infty} \frac{\sin^k x}{\cos^{k+1} x} \frac{\sin^k y}{\cos^{k+1} y},$$

which corresponds to the expansion

$$(8) \qquad f(z) = \sum_{n=0}^{\infty} S_n z^n = \cfrac{1}{1 - \cfrac{1^2 \cdot z^2}{1 - \cfrac{2^2 \cdot z^2}{1 - \cfrac{3^2 \cdot z^2}{\ddots}}}},$$

with $S_n = n![z^n]\sec(z)$ a secant number.

4. Combinatorics of continued fractions

Continued fractions like (3), (6), (7), (8) with such regular patterns cannot leave a combinatorialist indifferent as the Taylor coefficients count unconstrained permutations, set partitions, involutions, and alternating permutations respectively. At the same time, the continued fractions have coefficients given by simple integral laws.

A basic theorem due to Touchard [13], Good [3], Lenard [6], Szekeres [12], Jackson [5], Flajolet [2], and Read [9] expresses a general connection between J- and S-fractions on the one hand, lattice walks in $\mathbb{Z} \times \mathbb{Z}$ on the other hand. It is no accident that it arises naturally in calculations of random walk probabilities [3], and it is strongly connected with combinatorial configurations equivalent to lattice paths, like chord systems [9, 13]. The original inspiration for [2], is the dynamic analysis of data structures ("histories" introduced by Françon), while Jackson [5] and Lenard [6] draw their motivations from certain one-dimensional models of statistical physics (the Ising model and electrostatics, respectively). Szekeres [12], on the other hand, started from scattered observations of Ramanujan regarding general continued fractions. (Apart from [2], a description of the basic theory may be found in the book by Jackson and Goulden [4, Ch. V].)

Consider the path on the integer lattice $\mathbb{Z} \times \mathbb{Z}$ made of three types of steps

$$\textit{ascents } \binom{1}{1}, \ \textit{descents } \binom{1}{-1}, \text{ and } \textit{levels } \binom{1}{0},$$

that start at the origin, finish at altitude 0, and are constrained to stay in the upper-right quarter plane. Each walk can be encoded multiplicatively by a (noncommutative) monomial defined by associating

$$1, \lambda_j, \kappa_j,$$

to a step of type ascent, descent with starting altitude equal to j, level with altitude j, respectively. The walk polynomial W_n is the sum of all monomial encodings of all walks made of n steps.

THEOREM 1 (TGLSJFR). *The universal J-fraction is the generating function of the set of walk polynomials*

$$\cfrac{1}{1-\kappa_0\cdot z-\cfrac{\lambda_1\cdot z^2}{1-\kappa_1\cdot z-\cfrac{\lambda_2\cdot z^2}{1-\kappa_2\cdot z-\cfrac{\lambda_3\cdot z^2}{\ddots}}}}=\sum_{n=0}^{\infty}W_n z^n \equiv 1+\kappa_0 z+(\kappa_0+\lambda_1)z^2+\cdots.$$

This theorem was in effect used to establish combinatorially all the continued fraction expansions (3), (6), (7), (8), by appealing in an essential way to combinatorial bijections of Françon and Viennot. Conversely, any enumeration problem that can be reduced to a weighted lattice path counting leads to a continued fraction expression for the corresponding ordinary generating function. From there, the whole arsenal of special function identities can then be employed.

An illustrative example is the counting of “coin fountains” by Odlyzko and Wilf. An n-fountain is an arrangement of coins in rows such that the first row has no gaps and each coin in a higher row touches exactly two coins in the next lower row (see Example 10.7 of [7]). The contour of an n-fountain clearly resembles a legal lattice path, and from the TGLSJFR theorem (by adapting the weights), one gets for the ordinary generating function of fountains the representation

$$F(q)=\sum_{n=0}^{\infty}F_n q^n=\cfrac{1}{1-\cfrac{q}{1-\cfrac{q^2}{1-\cfrac{q^3}{\ddots}}}}.$$

The continued fraction is expressible in terms of q-exponentials and an analysis of its dominant polar singularity furnishes the very precise asymptotic formula:

$$F_n \sim C\cdot A^n \qquad \text{with} \qquad C\simeq 0.31236, \quad A\simeq 1.73566.$$

A comparable process has led to the solution of several counting problems like shared communication networks (in terms of Hermite polynomials, by Lagarias and Odlyzko), urn models of the Ehrenfest type (by Goulden and Jackson), correlated Gaussian variables (by Odlyzko *et al.*), non-overlapping partitions (by Flajolet and Schott), etc.

In any case, a continued fraction representation is usually the centre of a rich cluster of identities involving a specific class of orthogonal polynomials, Hankel matrices, and Stieltjes matrices.

5. Some special combinatorial fractions

Because of space constraints, we can only allude to some of the other combinatorial continued fractions discussed by Roblet in his lecture. See [10] for details.

Permutations. The continued fraction

$$\cfrac{1}{1 - f\cdot z - \cfrac{tpq[1]_q^2\cdot z^2}{1 - q(fq + (f+r)[1]_q)\cdot z - \cfrac{tpq^3[2]_q^2\cdot z^2}{1 - q^2(fq^2 + (f+r)[2]_q)\cdot z - \cfrac{tpq^5[3]_q^2\cdot z^2}{\ddots}}}}$$

is the ordinary generating function of permutations counted according to inversions (marked by q), and the cyclic structure: fixed points (marked by f), peaks of cycles [*i.e.*, $\sigma^{-1}(i) < i > \sigma(i)$] (marked by p), troughs [*i.e.*, $\sigma^{-1}(i) > i < \sigma(i)$] (marked by t), double rises (marked by r), and double falls (marked by f). This development relates to a bijection of Biane and it provides a 6-parameter statistic on permutations. (There, as usual $[k]_q = 1 + q + \cdots + q^{k-1}$.)

Another bijection of Roblet and Viennot yields a trivariate statistics on permutations by means of the continued fraction

$$\cfrac{1}{1 - (ab - [a;q]_1)\cdot z - \cfrac{[a;q]_1\cdot z}{1 - (bq - [a;q]_2)\cdot z - \cfrac{[a;q]_2\cdot z}{1 - (bq^2 - [a;q]_3)\cdot z - \cfrac{[a;q]_3\cdot z}{\ddots}}}},$$

with a marking right-to-left minima and b marking left-to-right maxima, with q again for inversions. (There, $[a;q] = a + q + q^2 + \cdots + q^{k-1}$.) Notice that the above fraction is not a J-fraction, as the numerators involve z instead of z^2. It is known as a T-fraction (for Thron). Obtaining such continued fractions require changing the notion of legal lattice path, and it is one of the major contributions of [10] to develop a systematic theory of such fractions.

Polyominoes. Parallelogram polyominoes can be encoded by various types of lattice paths (Delest, Viennot, Fédou). Roblet gives a T-fraction representation for the joint statistics of height, width, area, and perimeter.

6. Padé approximants and T-fractions

Roblet's thesis also proposes a general theory of continued fraction expansions that differ from the basic J- or S-type. We have already encountered the examples of T-fractions.

An important class of rational approximants is that of Padé fractions. A general combinatorial approach for them is given in [10]. This necessitates appreciably modifying the notion of legal paths. A valuable result is then to explain combinatorially the possible degeneracies in the Padé table, while simply interpreting the basic algebraic identities of the theory.

From his combinatorial interpretations, Roblet is able to deduce systematic (and novel) algorithms for the expansion of a power series into various types of continued fractions. The design is based on a generalization of the Stieltjes matrix that in the classical case is related to the ϕ_k in the Stieltjes-Rogers addition formula and combinatorially to path terminating at an altitude different from 0. The algorithms so obtained have many desirable features: they are "incremental" (a useful feature for computer algebra applications) and of low complexity, consuming space $O(n)$ and time $O(n^2)$. Again, we have to refer to Roblet's thesis for details.

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