Introduction to q-calculus

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[summary by Xavier Gourdon]

Abstract

Many mathematical formulæ can be generalised by adding a new parameter q, leading to what is called a q-analogue, because the original formula can be obtained as the limit when q tends to 1. We present here a combinatorial introduction to the q-calculus.

1. Partitions and words

1.1. Partitions. A partition λ is a decreasing sequence $(\lambda_1, \ldots, \lambda_k)$ of positive integers: $\lambda_i \in \mathbb{N}^*$ and $\lambda_i \geq \lambda_{i+1}$ for all i. The length of λ is $\ell(\lambda) = k$, its height is $|\lambda| = \sum_{i=1}^{k} \lambda_i$. If $|\lambda| = n$, we say that λ is a partition of n. For all $\ell, m \in \mathbb{N}$, let

$$P(\ell, m) = \{\lambda : \ \ell(\lambda) \leq \ell \quad \text{and} \quad \lambda_1 \leq m\}.$$

It is possible to determine a partition λ from the numbers $m_i = \text{Card}\{j : \ \lambda_j = i\}$ denoting the multiplicity of i in λ. In this way, the partition λ can be written as $\lambda = (1^{m_1} 2^{m_2} \cdots)$.

A nice way to represent a partition λ is to use its Ferrers diagram

$$D_\lambda = \{(i,j) \in \mathbb{Z}^2 : \ 1 \leq i \leq \lambda_j \quad \text{and} \quad 1 \leq j \leq \ell(\lambda)\}.$$

The number $P(\ell, m)$ can be viewed as $P(\ell, m) = \{\lambda : \ D_\lambda \subset (m^\ell)\}$. The conjugate partition of λ is the partition λ' whose Ferrers diagram is symmetric from D_λ with respect to the first bisecting line. We have $|\lambda'| = |\lambda|$ and $(\lambda')' = \lambda$.

1.2. Gaussian polynomials. Ferrers diagrams enable to establish a correspondence between partitions of $P(\ell, m)$ and paths joining $(0, \ell)$ to $(m, 0)$ with the steps $(0, -1)$ and $(1, 0)$. Such paths have $m + \ell$ steps (m horizontal and ℓ vertical) so Card $P(\ell, m) = \binom{m+\ell}{m}$. To take into account the height in this statistic, we introduce its generating function with respect to a new variable q. We have

$$(1) \qquad \sum_{\lambda \in P(\ell, m)} q^{|\lambda|} = \frac{(q)_{m+\ell}}{(q)_m (q)_\ell} \quad \text{where} \quad \begin{cases} (q)_k = \prod_{i=1}^{k} (1 - q^i) & k \geq 1, \\ (q)_0 = 1. \end{cases}$$

Letting $q \to 1$ in this identity, we find again Card $P(\ell, m) = \binom{m+\ell}{m}$. This motivates the definition of a q-analogue of the binomial coefficients, denoted by

$$\begin{bmatrix} m + \ell \\ \ell \end{bmatrix} = \frac{(q)_{m+\ell}}{(q)_m (q)_\ell},$$

and called Gaussian polynomials. They satisfy several q-properties like Pascal recurrences or symmetry.

By letting $\ell \to \infty$ in identity (1), we get

$$\sum_{\lambda:\ \lambda_1 \le m} q^{|\lambda|} = \frac{1}{(q)_m} = \sum_{\lambda:\ \ell(\lambda)\le m} q^{|\lambda|}. \tag{2}$$

This last equality is obtained from the conjugate partitions. Then letting $m \to \infty$, we find

$$\sum_{\lambda} q^{|\lambda|} = \sum_{n\ge 0} p(n) q^n = \frac{1}{(q)_\infty}, \tag{3}$$

where $p(n)$ is the total number of partitions of n.

1.3. Infinite products. Formula (2) can be refined by introducing a new variable x. More precisely, denoting

$$(x)_\infty = \prod_{i=0}^{\infty}(1 - xq^i),$$

we have the identity

$$\frac{1}{(x)_\infty} = \sum_{m_0, m_1, \dots} \left(\prod_{i=0}^{\infty} x^{m_i} q^{i m_i} \right) = \sum_{\ell\ge 0} x^\ell \sum_{\lambda:\ \ell(\lambda)\le \ell} q^{|\lambda|} = \sum_{\ell \ge 0} \frac{x^\ell}{(q)_\ell}.$$

In the same vein, by expanding $(-x)_\infty$ we have

$$(-x)_\infty = \sum_{\ell\ge 0} \frac{q^{\binom{\ell}{2}}}{(q)_\ell} x^\ell.$$

These two identities are sometimes called Euler identities.

Jacobi identity. The triple product identity of Jacobi is

$$(q)_\infty (x)_\infty (qx^{-1})_\infty = \sum_{n\in\mathbb{Z}} (-1)^n x^n q^{\binom{n}{2}}.$$

As a corollary, we have the formulæ

$$(q)_\infty = \sum_{n\in\mathbb{Z}} (-1)^n q^{n(3n+1)/2} \tag{4}$$

$$(q)_\infty^3 = \sum_{n\in\mathbb{N}} (-1)^n (2n+1) q^{n(n+1)/2}. \tag{5}$$

The first one is Euler's pentagonal number theorem, and can be used with (3) to establish several congruences relations satisfied by the partition numbers $p(n)$.

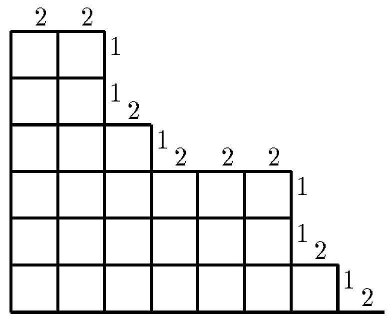

FIGURE 1. With $\ell = 6$ and $m = 9$, the partition $\lambda = (8, 7, 7, 4, 2, 2)$ is associated to the word $w = 221122122211212$.

2. Words

2.1. Correspondence between partitions and binary words. Consider $\lambda \in P(\ell, m)$. Its Ferrers diagram can be considered as a path joining points $(0, \ell)$ and $(m, 0)$ by m horizontal steps and ℓ vertical steps. We encode this path with a word on $\{1, 2\}^*$, associating a 1 for each vertical step, a 2 for each horizontal step (see figure 1). This construction defines a correspondence between $P(\ell, m)$ and $M(\ell, m)$, the words of $\{1, 2\}^*$ with ℓ "1" and m "2".

We define the number of inversions of a word w in $\{1, 2\}^*$ by

$$\operatorname{Inv} w = \operatorname{Card}\{(i, j) : \ 1 \le i < j \le \ell + m \quad \text{and} \quad 2 = w_i > w_j = 1\}. \tag{6}$$

We have $\operatorname{Inv} w = |\lambda|$, where w is the word obtained from λ by the correspondence, thus

$$\sum_{w \in M(\ell, m)} q^{\operatorname{Inv}(w)} = \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}.$$

Another interesting parameter is the major index defined by

$$\operatorname{Maj} w = \sum_{w_i > w_{i+1}} i, \tag{7}$$

and surprisingly, its generating function is the same as the one of Inv.

2.2. Statistics on words over n letters. The previous discussion finds a natural generalisation by considering $M(a_1, \dots, a_n)$, the set of words with n letters where the i-th letter appears exactly a_i times. The length of such a word w is $a_1 + \cdots + a_n$. The parameters $\operatorname{Inv} w$ and $\operatorname{Maj} w$ are defined as in (6) and (7). The Z-statistic (called like this because of Zeilberger work [2]) of a word w is defined as

$$z(w) = \sum_{1 \le i < j \le n} \operatorname{Maj} w_{i,j}$$

where $w_{i,j}$ is the word obtained from w by keeping only the i-th and the j-th letter. These parameters satisfy

$$\sum_{w \in M(a_1, \dots, a_n)} q^{\operatorname{Inv}(w)} = \sum_{w \in M(a_1, \dots, a_n)} q^{\operatorname{Maj}(w)} = \sum_{w \in M(a_1, \dots, a_n)} q^{z(w)} = \begin{bmatrix} a_1 + \cdots + a_n \\ a_1, \dots, a_n \end{bmatrix} := \frac{(q)_{a_1 + \cdots + a_n}}{(q)_{a_1} \cdots (q)_{a_n}},$$

providing a q-analogue of multinomial coefficients.

3. Basic hypergeometric functions

We use the notations

$$(a)_\infty = (a;q)_\infty = \prod_{i=0}^{\infty}(1 - aq^i), \qquad (a)_n = (a;q)_n = \frac{(a)_\infty}{(aq^n)_\infty}$$

and we define the basic hypergeometric series as

$$ {}_r\phi_s\left(\begin{matrix} \alpha_1, \ldots, \alpha_r \\ \beta_1, \ldots, \beta_s \end{matrix} ; x \right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(q)_n(\beta_1)_n \cdots (\beta_r)_n} x^n.$$

The $q \to 1$ limit in this expression leads to a classical hypergeometric series, thus we have defined a q-analogue of hypergeometric series. A good survey of basic hypergeometric series can be found in [3].

3.1. The q-binomial theorem. The relation $(1 - x)\,{}_1\phi_0(a;x) = (1 - ax)\,{}_1\phi_0(a;qx)$ together with ${}_1\phi_0(a;0) = 1$ leads to the q-binomial theorem:

$$ {}_1\phi_0(a;x) = \frac{(ax)_\infty}{(x)_\infty}. \tag{8}$$

By setting $a = q^{-n}$ then $x \to -xq^{-n}$, we deduce

$$\prod_{i=0}^{n-1}(1 + q^i x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} x^k.$$

When $q \to 1$, this leads to the classical binomial theorem.

3.2. Heine transforms. Like classical hypergeometric functions, the basic hypergeometric functions satisfy several identities. A first family is the Heine transforms:

$$ {}_2\phi_1\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix} ; x \right) = \frac{(\beta)_\infty(\alpha x)_\infty}{(\gamma)_\infty(x)_\infty}\; {}_2\phi_1\left(\begin{matrix} \gamma/\beta, x \\ \alpha x \end{matrix} ; \beta \right)$$

$$ {}_2\phi_1\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix} ; x \right) = \frac{\left(\frac{\alpha\beta}{\gamma} x\right)_\infty}{(x)_\infty}\; {}_2\phi_1\left(\begin{matrix} \gamma/\alpha, \gamma/\beta \\ \gamma \end{matrix} ; \frac{\alpha\beta}{\gamma} x \right).$$

3.3. Pfaff-Saalschütz q-theorem. This result applies to functions of the type ${}_3\phi_2$. For all non-negative integer n, we have

$$ {}_3\phi_2\left(\begin{matrix} a, b, q^{-n} \\ c, \frac{ab}{c} q^{1-n} \end{matrix} ; q \right) = \frac{(c/a)_n(c/b)_n}{(c)_n(c/ab)_n}. \tag{9}$$

There exists several equivalent forms of this theorem. By letting $n \to +\infty$ in (9), we obtain the Gauss q-theorem

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix} ; \frac{c}{ab} \right) = \frac{(c/a)_\infty(c/b)_\infty}{(c)_\infty(c/ab)_\infty}.$$

Another corollary of the Pfaff-Saalschütz q-theorem is the q-formula of Chu-Vandermonde, obtained by setting $a = q^{n+1}$, $b = q^{-k}$ and $c = q^{m+1}$ in (9)

$$\sum_{i=0}^{k} (-1)^i q^{i(i-1)/2 + (m-n)i} \begin{bmatrix} n+i \\ i \end{bmatrix} \begin{bmatrix} m+k \\ k-i \end{bmatrix} = \begin{bmatrix} k+m-n-1 \\ k \end{bmatrix}.$$

There exists several generalizations of the Pfaff-Saalschütz q-theorem. One is called the Dougall q-theorem, it applies to functions of the type ${}_8\phi_7$.

4. q-analogues of usual tools

4.1. q-derivative. The q-derivative of a function f is defined as

$$\delta_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}.$$

The formulæ of the classical derivative have their q-analogues with respect to the q-derivative.

4.2. q-integration. The function $g(t) = \int_0^t f(x)\, d_q x$ must satisfy $\delta_q g = f$, so

$$\begin{aligned} g(t) - g(qt) &= t(1-q)f(t) \\ g(qt) - g(q^2t) &= qt(1-q)f(qt) \\ \cdots &= \cdots \end{aligned}$$

thus $g(t) = g(t) - g(0) = \sum_{n\geq 0} q^n t(1-q)f(q^n t)$, and we define

$$\int_0^t f(x)\, d_q x = t(1-q)\sum_{n\geq 0} q^n f(q^n t).$$

Like the classical integral, there exists a q-formula of integration by parts. There are several ways of defining an improper integral, for example

$$\int_0^{+\infty} f(t)\, d_q t = (1-q)\sum_{n\in\mathbb{Z}} q^n f(q^n) \quad \text{and} \quad \int_0^{+\infty} f(t)\, d_q t = \int_0^{1/(1-q)} f(t)\, d_q t$$

are q-analogues of $\int_0^{+\infty} f(t)\, dt$.

4.3. q-differential equations. The q-differential equation $\delta_q f(t) = f(t)$ admits the solution

$$f(t) = \frac{f(qt)}{1 - t(1-q)} = \cdots = \frac{f(0)}{(t(1-q))_\infty},$$

thus the solution f with $f(0) = 1$ is

$$e_q(t) = \frac{1}{(t(1-q))_\infty} = \sum_{n\geq 0} \frac{(1-q)^n}{(q)_n t^n},$$

(the last identity is obtained from the q-binomial theorem (8) with $a = 0$ and $x = t(1-q)$) providing a q-analogue of the expansion of $\exp(t)$.

As for the q-differential equation $\delta_q f(t) = f(qt)$, the solution which takes the value 1 at 0 is

$$E_q(t) = (-t(1-q))_\infty = \sum_{n\geq 0} q^{\binom{n}{2}} \frac{(1-q)^n}{(q)_n} t^n,$$

the last identity being a consequence of the q-binomial theorem applied with $a = -t(1-q)/x$ and $x \to 0$. This second q-analogue of the expansion of $\exp(t)$ satisfy the obvious relation $e_q(t)E_q(-t) = 1$. Nevertheless, there does not exist any simple relation between $e_q(x)e_q(y)$, $E_q(x)E_q(y)$ and

$e_q(x+y)$, $E_q(x+y)$. A q-analogue of the relation $\exp(x+y) = \exp(x)\exp(y)$ is given by the formula

$$e_q(x)E_q(y) = \sum_{n=0}^{+\infty} \frac{\prod_{k=0}^{n-1}(x+q^k y)}{\prod_{k=1}^{n} \frac{1-q^k}{1-q}},$$

obtained from the q-binomial theorem with $a = -y/x$ and $x = x(1-q)$.

4.4. The q-gamma function. The q-gamma function is defined as

$$\Gamma_q(s) = \frac{(q)_\infty}{(q^s)_\infty}(1-q)^{1-s}, \qquad s \in \mathbb{C} \setminus \{0, -1, -2, \dots\},$$

which tends to $\Gamma(s)$ as $q \to 1$. The functional equation of Γ_q is

$$\Gamma_q(s+1) = \frac{1-q^s}{1-q}\Gamma_q(s),$$

and since $\Gamma_q(1) = 1$, we have for all positive integer n

$$\Gamma_q(n+1) = \prod_{k=1}^{n} \frac{1-q^k}{1-q} = \frac{(q)_n}{(1-q)^n},$$

which is a q-analogue of $\Gamma(n+1) = n!$. The function $\log\Gamma_q(x)$ is convex for $x > 0$. An integral representation of Γ_q is

$$\Gamma_q(s) = \int_0^{1/(1-q)} t^{s-1} E_q(-qt)\, d_q t.$$

There also exists a q-analogue of the Gauss duplication formula.

4.5. The q-beta function. An equivalent form of the q-binomial theorem is

$$\int_0^1 t^{x-1} \frac{(qt)_\infty}{(q^y t)_\infty}\, d_q t = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}, \qquad (\Re(x) > 0).$$

This expression can be used to define the most well known and the most useful q-analogue of the beta function.

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