Eulerian Calculus and Transformations of Rearrangements

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[summary by Dominique Gouyou-Beauchamps]

1. Introduction

The purpose of this talk is to study the behaviour of several classical statistics on words, such as the number of descents, the number of excedances, the major index, when the strict inequalities required in their definitions are relaxed to include some equalities[1]. Those new statistics are defined on classes of words with repetitions.

Let X^* be the free monoid generated by a totally ordered alphabet X that for convenience we take as being the subset $[r] = \{1, 2, \ldots, r\}$ $(r \geq 1)$ of the positive integers. X is a fixed non-empty set on which a total ordering D is defined. If $x, y \in X$ and $(x, y) \in D$ we write $x <_D y$ or simply $x < y$ if no confusion can arise. D need not be the standard ordering on $[r]$. We also have a fixed integer, k, such that $0 \leq k \leq r$ and $j = r - k$. The letters $1, \ldots, j$ will be called *small* and the letters $j+1, \ldots, r$ *large*. We say that D is *compatible* with k if, for all x large and y small, we have $x > y$. We also introduce a small letter $*$ which is greater than any small letter of X.

A word w' is said to be a *rearrangement* of the word $w = x_1 x_2 \cdots x_m$ if it can be obtained from w by permuting the letters $x_1, x_2, \ldots, x_m$ in some order. The set of all the rearrangements of a word w will be denoted $C(w)$. Such a set necessarily contains a unique word $v = y_1 y_2 \cdots y_m$ whose letter are in non-decreasing order: $y_1 \leq y_2 \leq \cdots \leq y_m$. It will be convenient to denote $\overline{w}$ the unique non-decreasing word in the class $C(w)$.

Let $w = x_1 x_2 \cdots x_m$ be a word and let $\overline{w} = v = y_1 y_2 \cdots y_m$ be its non-decreasing rearrangement. The *number of excedances*, $\operatorname{exc} w$, and the *number of descents*, $\operatorname{des} w$, of the word w are classically defined as

$$\operatorname{exc} w = \#\{i : 1 \leq i \leq m, x_i > y_i\},$$
$$\operatorname{des} w = \#\{i : 1 \leq i \leq m-1, x_i > x_{i+1}\},$$

while the *major index*, $\operatorname{maj} w$, is the sum of the i's such that $1 \leq i \leq m-1$ and $x_i > x_{i+1}$.

MacMahon [8, p. 186] proved that for each rearrangement class $C(v)$ and each integer j there are as many words $w \in C(v)$ such that $\operatorname{exc} w = j$ as there are words $w' \in C(v)$ such that $\operatorname{des} w' = j$.

Let $\mathbf{c} = (c_1, \ldots, c_j)$ and $\mathbf{d} = (d_1, \ldots, d_k)$ be two vectors with positive integer components. Also let $c = c_1 + \cdots + c_j$, $d = d_1 + \cdots + d_k$ and $c + d = m$. The class of all $m!/(c_1! \cdots c_j! d_1! \cdots d_k!)$ rearrangements of the word $1^{c_1} \cdots j^{c_j} (j+1)^{d_1} \cdots r^{d_k}$ will be denoted by $R(\mathbf{c}, \mathbf{d})$ or by $C(v)$ where v is a given word in $R(\mathbf{c}, \mathbf{d})$.

Let $w = x_1 x_2 \cdots x_m$ be a word and let $\overline{w} = v = y_1 y_2 \cdots y_m$ be its non-decreasing rearrangement (with respect to a given ordering D). We say that the word w has a *k-excedance* at i $(1 \leq i \leq m)$,

[1]The original articles by J. Clarke and D. Foata can be found in [2, 3, 4].

if either $x_i > y_i$, or $x_i = y_i$ and x_i large. We also say that w has a k-*descent* at i $(1 \le i \le m)$, if either $x_i > x_{i+1}$, or $x_i = x_{i+1}$ and x_i large (by convention, $x_{m+1} = *$). The number of k-excedances and k-descents of a word w are denoted by $\operatorname{exc}_k w$ and $\operatorname{des}_k w$. The k-*major index*, $\operatorname{maj}_k w$, is the sum of all i's $(1 \le i \le m)$ such that i is a k-descent.

In [2] R. J. Clarke and D. Foata showed that for each ordering D compatible with $k \ge 0$ the statistics "des_k" and "exc_k" were equidistributed on each rearrangement class $R(\mathbf{c}, \mathbf{d})$. Actually, they constructed a bijection Φ_k of each rearrangement class $R(\mathbf{c}, \mathbf{d})$ onto itself that satisfied $\operatorname{des}_k w = \operatorname{exc}_k \Phi_k(w)$, identically. Hence for each rearrangement class $R(\mathbf{c}, \mathbf{d})$ the generating polynomials $\sum_w t^{\operatorname{des}_k w}$ and $\sum_w t^{\operatorname{exc}_k w}$ $(w \in R(\mathbf{c}, \mathbf{d}))$ are equal. Let $A_{\mathbf{c},\mathbf{d}}(t)$ be their common value. It was also shown that the generating function for those polynomials could be expressed as

$$\sum_{\mathbf{c},\mathbf{d}} \frac{\mathbf{u}^{\mathbf{c}}\mathbf{v}^{\mathbf{d}}}{(1-t)^{c+d+1}} A_{\mathbf{c},\mathbf{d}}(t) = \sum_{s \ge 0} t^s \frac{(1+v_1)^s \cdots (1+v_k)^s}{(1-u_1)^{s+1} \cdots (1-u_j)^{s+1}}, \tag{1}$$

where $\mathbf{u}^{\mathbf{c}} = u_1^{c_1} \cdots u_j^{c_j}$ and $\mathbf{v}^{\mathbf{d}} = v_1^{d_1} \cdots v_k^{d_k}$.

As usual, let $(a;q)_n$ denote the q-ascending factorial

$$(a;q)_n = \begin{cases} 1 & \text{if } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{if } n \ge 1. \end{cases}$$

Then, a natural q-analogue of (1) can read

$$\sum_{\mathbf{c},\mathbf{d}} \frac{\mathbf{u}^{\mathbf{c}}\mathbf{v}^{\mathbf{d}}}{(t;q)_{c+d+1}} A_{\mathbf{c},\mathbf{d}}(t,q) = \sum_{s \ge 0} t^s \frac{(-qv_1;q)_s \cdots (-qv_k;q)_s}{(u_1;q)_{s+1} \cdots (u_j;q)_{s+1}}. \tag{2}$$

The motivation of the talk is to extend Han's construction [6] to weighted words. This consists, first, in finding an appropriate extension "den_k" of the Denert statistic "den" [5], defined by Han, then, of constructing an explicit bijection ρ of each rearrangement class $R(\mathbf{c}, \mathbf{d})$ onto itself such that the equality over the bivariate statistics $(\operatorname{des}_k, \operatorname{maj}_k)(w) = (\operatorname{exc}_k, \operatorname{den}_k)\rho(w)$ holds identically.

2. The "den_k" statistic

Let $S = \{1, \ldots, j\}$ be the set of small elements of X and let $L = \{j+1, \ldots, r\}$ be the set of large elements of X. Let $s_{\max}$ be the largest small letter of X (under the ordering D). Besides the small letter $*$ that satisfies $s_{\max} <_D * <_D b$ for any letter b greater than $s_{\max}$, we also adjoin to X a large letter ∞ that is greater than every letter of X. Define X^+ to be $X \cup \{*, \infty\}$. Similarly, $L^+ = L \cup \{\infty\}$ and $S^+ = S \cup \{*\}$.

Let a and b be elements of X^+. Then we define the *cyclic interval* $\rrbracket a, b \rrbracket$ by

$$\rrbracket a, b \rrbracket = \begin{cases} (a, b] & \text{if } a \le b, \\ X^+ \backslash (b, a] & \text{otherwise.} \end{cases} \tag{3}$$

Thus $\rrbracket a, a \rrbracket = \emptyset$. Further, we define $\rrbracket a, b \rrbracket_k$ by

$$\rrbracket a, b \rrbracket_k = \begin{cases} \rrbracket a, b \rrbracket & \text{if } a, b \in S^+, \\ \rrbracket a, b \rrbracket \cup a & \text{if } a \in L^+,\ b \in S^+, \\ \rrbracket a, b \rrbracket \setminus b & \text{if } a \in S^+,\ b \in L^+, \\ \rrbracket a, b \rrbracket \cup a \setminus b & \text{if } a, b \in L^+,\ a \ne b, \\ X^+ & \text{if } a = b \in L^+. \end{cases} \tag{4}$$

The elements of X^+ can be visualized as points on a circle (or a square!) as shown on Fig. 1. The k-cyclic intervals $]\!]a,b]\!]_k$ must be read counterclockwise. The path $\vdash\!\!\rightarrow$ on the S^+-part shows that whenever a is small, the interval $]\!]a,b]\!]_k$ is of the form "$(a,\ldots$" (or "$]a,\ldots$" in the French notation) so that $a \notin]\!]a,b]\!]_k$. On the contrary, the path $\dashv\!\!\rightarrow$ on the L^+-part shows that $]\!]a,b]\!]_k = [a,\ldots$ and so $a \in]\!]a,b]\!]_k$ whenever a is large; but $]\!]a,b]\!]_k = \ldots, b)$ (or $\ldots, b[$) whenever b is large and $b \notin]\!]a,b]\!]_k$. When D is compatible with k, the small letters lie between ∞ and $*$, and the large ones between $*$ and ∞, still reading the square counterclockwise; also $]\!]*,\infty]\!]_k = L$.

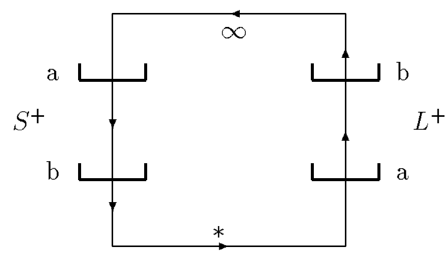

Fig. 1

Let $w = x_1x_2\cdots x_m$ be a word on the letters in X. Put $x_{m+1} = *$, $x_{m+2} = \infty$. For $i = 1,\ldots,m+2$ let $\mathrm{Fact}_i w$ be the left factor $x_1x_2\cdots x_{i-1}$ of w and for each subset B of X let $\mathrm{Fact}_i w \cap B$ be the subword of $\mathrm{Fact}_i w$ consisting only of those letters of $\mathrm{Fact}_i w$ that are in B. Furthermore, let $|\mathrm{Fact}_i w \cap B|$ denote the length of that subword.

Now let $\overline{w} = y_1y_2\cdots y_m$ the non-decreasing rearrangement of a word $w = x_1x_2\cdots x_m$. The den_k-coding of w is defined to be the sequence $(s_i)_{1\le i\le m+1}$, where

(5) $$s_i = \begin{cases} |\,\mathrm{Fact}_i\, w \cap]\!]x_i, y_i]\!]_k| & \text{if } 1 \le i \le m,\\ |w \cap L| & \text{if } i = m+1,\end{cases}$$

and the statistic $\mathrm{den}_k\, w$ to be

(6) $$\mathrm{den}_k\, w = \sum_{i=1}^{m+1} s_i.$$

Theorem 1. *Let v be a fixed word in X^* and let D and E be total orderings on $X = [r]$. Assume that both D and E are compatible with k. Then there is a bijection μ on $C(v) = R(\mathbf{c},\mathbf{d})$ onto itself such that for all $w \in C(v)$,*

$$(\mathrm{des}_{k,D}, \mathrm{maj}_{k,D})w = (\mathrm{des}_{k,E}, \mathrm{maj}_{k,E})\mu(w).$$

Theorem 2. *Let v be a fixed word in X^* and let D be an ordering on X compatible with k. Then there is a bijection ρ on $C(v)$ onto itself such that for all $w \in C(v)$,*

$$(\mathrm{des}_k, \mathrm{maj}_k)w = (\mathrm{exc}_k, \mathrm{den}_k)\rho(w).$$

Theorem 3. *Let v be a fixed word in X^* and let D and E be total orderings on $X = [r]$. Then there is a bijection δ on $C(v) = R(\mathbf{c},\mathbf{d})$ onto itself such that for all $w \in C(v)$,*

$$(\mathrm{exc}_{k,D}, \mathrm{den}_{k,D})w = (\mathrm{exc}_{k,E}, \mathrm{den}_{k,E})\delta(w).$$

Now we calculate the distribution of $(\mathrm{des}_k, \mathrm{maj}_k)$. Let

$$A_{\mathbf{c},\mathbf{d}}(t,q) = \sum_w t^{\mathrm{des}_k w} q^{\mathrm{maj}_k w} \qquad (w \in R(\mathbf{c},\mathbf{d}))$$

be the generating function for the pair $(\mathrm{des}_k, \mathrm{maj}_k)$ over the class $R(\mathbf{c}, \mathbf{d})$.

THEOREM 4. *The factorial generating function for the polynomials $A_{\mathbf{c},\mathbf{d}}(t,q)$ satisfies (2).*

3. The Han transposition

In this section we describe the "Han transposition", a way of manipulating biwords that preserves the statistics "den" and "exc".

A *biword* is a two rowed matrix $\alpha = \binom{u}{w}$, where u and v are words in X^* of the same length. The biword α is called a *circuit* if u is a rearrangement of w. A circuit $\alpha = \binom{y_1 y_2 \cdots y_m}{x_1 x_2 \cdots x_m}$ is called a *cycle*, if $y_m = x_1$ and $y_i = x_{i+1}$ for $i = 1, \ldots, m-1$.

Let $x, y, a, b \in X \cup \{*\}$. Then a and b are *neighbours* with respect to (x, y) if both a and b are in $]\!]a, b]\!]_k$ or neither in $]\!]a, b]\!]_k$. Otherwise, a and b are strangers with respect to (x, y).

Consider the biword $\binom{u}{w}$, where $u = y_1 y_2 \cdots y_m$ and $w = x_1 x_2 \cdots x_m$. An ordering D of X being given, we define

$$\mathrm{exc}_k \begin{pmatrix} u \\ w \end{pmatrix} = |\{i : 1 \le i \le m \quad \text{and} \quad x_i > y_i \quad \text{or} \quad x_i = y_i \in L^+\}|;$$

$$\mathrm{den}_k \begin{pmatrix} u \\ w \end{pmatrix} = \sum_{i=1}^{m} |\mathrm{Fact}_i w \cap]\!]x_i, y_i]\!]_k|.$$

If $\overline{w}$ is the non-decreasing rearrangement of w, then clearly $\mathrm{exc}_k \binom{\overline{w}}{w} = \mathrm{exc}_k\, w$; and by (5) and (6) $\mathrm{den}_k\, w = \mathrm{den}_k \binom{\overline{w}}{w} + |w \cap L|$. Note that if D is compatible with k, $\mathrm{den}_k\, w = \mathrm{den}_k \binom{\overline{w}\infty}{w*}$.

Let $\binom{xy}{ab}$ be a biword of length two. Following Han [6] we define the Han transposition T by

$$T\begin{pmatrix} xy \\ ab \end{pmatrix} = \begin{cases} \binom{yx}{ab} & \text{if } a \text{ and } b \text{ are neighbours,} \\ \binom{yx}{ba} & \text{if } a \text{ and } b \text{ are strangers.} \end{cases} \tag{7}$$

If $\alpha = \binom{u}{w} = \binom{y_1 y_2 \cdots y_m}{x_1 x_2 \cdots x_m}$ is a biword of length m and $1 \le i < m$, we define $T_i \alpha$ to be the biword obtained when the biword $\beta = \binom{y_i y_{i+1}}{x_i x_{i+1}}$ consisting of the i-th and $(i+1)$th columns of α is replaced by $T\beta$.

LEMMA 1. *Let $\alpha = \binom{u}{w}$ be a biword of length m and let $1 \le i < m$. Then*

$$(\mathrm{exc}_k, \mathrm{den}_k) T_i \alpha = (\mathrm{exc}_k, \mathrm{den}_k) \alpha.$$

Let z_1, z_2 be two distinct letters of $X \cup \{*\}$ and v be a word of length $m-1$ in the alphabet $X \cup \{*\} \setminus \{z_2\}$. Denote by $\mathcal{C}(v, z_1, z_2)$ the set of all biwords $\alpha = \binom{u}{w}$, where u is the non-decreasing rearrangement of vz_2 and w is any rearrangement of vz_1. Thus u has one occurrence of z_2, while w has none. However the occurrences of the other letters are the same, except for z_1 that occurs one more time in w than in u.

If z_2 is the i-th letter in the word u, the product $T_{m-1} \cdots T_{i+1} T_i$ will transform α into a biword of the form $\alpha' = \binom{u' z_2}{w' y_1}$. Then, either u' has no occurrence of y_1, in which case $y_1 = z_1$ and w' must be a rearrangement of u', or y_1 does occur in u'. In the former case, define $T_{z_2}(\alpha) = \binom{u' z_2}{w' z_1}$. In the latter case, the rightmost occurrence of y_1 in u' is, say, its i'-th letter. Then the product $T_{m-2} \cdots T_{i'+1} T_{i'}$ transforms α' into a biword of the form $\alpha'' = \binom{u'' y_1 z_2}{w'' y_2 y_1}$. Again, either u'' has no occurrence of y_2, in which case $y_2 = z_1$ and w'' must be a rearrangement of u'', or y_2 does occur in u''. In the former case, define $T_{z_2}(\alpha) = \binom{u'' y_1 z_2}{w'' z_1 y_1}$. In the latter case, we continue the same procedure as before by moving the rightmost occurrence of y_2 in u'' to the right of u''. After finitely many steps we reach a biword $\alpha^{(l)} = \binom{u^{(l)}y_{l-1}\cdots y_1z_2}{w^{(l)}y_l\cdots y_2y_1}$, where $u^{(l)}$ has no occurrence of y_l. Then necessarily $y_l = z_1$ and $w^{(l)}$ is a rearrangement of $u^{(l)}$. Note that $u^{(l)}$ may be empty. Define $u_1 = u^{(l)}$, $w_1 = w^{(l)}$, $v_1 = y_{l-1}\cdots y_2y_1$ and

$$T_{z_2}(\alpha) = \begin{pmatrix} u^{(l)}y_{l-1}\cdots y_1z_2 \\ w^{(l)}y_l\cdots y_2y_1 \end{pmatrix} = \begin{pmatrix} u_1v_1z_2 \\ w_1z_1v_1 \end{pmatrix}. \tag{8}$$

Thus for each α in $\mathcal{C}(v, z_1, z_2)$ there is a well-defined product of Han transpositions that maps α onto a biword of the form $\binom{u_1v_1z_2}{w_1z_1v_1}$, where u_1 is the non-decreasing rearrangement of w_1 with no occurrence of z_1. Denote by $\mathcal{D}(v, z_1, z_2)$ the set of biwords of the previous form $\binom{u_1v_1z_2}{w_1z_1v_1}$.

LEMMA 2. *The mapping* $T_{z_2} : \mathcal{C}(v, z_1, z_2) \to \mathcal{D}(v, z_1, z_2)$ *is a bijection.*

The inverse mapping applied to the biword $\binom{u_1v_1z_2}{w_1z_1v_1}$ in $\mathcal{D}(v, z_1, z_2)$ is derived by moving to the left, to the first position where the resulting word is non-decreasing, successively the first, the second, ..., the last letter z_2 of the word v_1z_2, the move being made by means of the Han transpositions as defined in (7). The inverse mapping is then independent of z_2 and will be denoted by T^{-1}.

4. The den-maj bijection

In this section we give the main tools for proving Theorem 2. We assume that the ordering D is compatible with k, and in fact that D is the standard ordering on $X = [r]$. If y is a letter and w a non-decreasing word, we will write $y < w$ (resp. $y \le w$), if y is less than (resp. less than or equal to) all the letters in w.

As for the first fundamental transformation described in Cartier and Foata [1] or in Lothaire [7, chap. 10] and Han's fundamental bijection [6] we need an appropriate *word factorization*. That factorization can be built as follows.

Let $z \in X \cup \{*\}$ and let v be a word in that alphabet, then the word zv is said to be *k-dominant*, if z is large and all letters in v are greater than or equal to z, or if z is small and small letters in v are less than or equal to z.

Every word in the alphabet $X \cup \{*\}$ has a unique factorization $(z_1v_1, z_2v_2, \ldots, z_nv_n)$ (the z_i's are letters and the v_i's words), called its *k-factorization*, having the following properties:

(1) $w = z_1v_1z_2v_2\cdots z_nv_n$;
(2) each factor z_iv_i is k-dominant $(1 \le i \le n)$;
(3) there exists an integer l $(1 \le l \le n)$ such that $z_1 > z_2 > \cdots > z_l > *$ and $z_{l+1} < z_{l+2} < \cdots < z_n \le *$.

The *k-factorization* of a word w may be obtained as follows: a letter z of w is called a *k-record*, if either z is large and all letters to the left of z are larger than z, or z is small and all small letters to the left of z are smaller than z. The *k-factorization* of w is then obtained by cutting w before each k-record.

The main property of the *k-factorization* on which our transformation is based is the following: let $(z_1v_1, z_2v_2, \ldots, z_nv_n)$ be the *k-factorization* of a word w. Then for each $i = 1, \ldots, n-1$ no letter in the left factor $(z_1v_1, \ldots, z_iv_i)$ is equal to z_{i+1} or is strictly between z_i and z_{i+1}. Let w_1 be any rearrangement of that factor; then the *k-factorization* of $(w_1z_{i+1}v_{i+1}\cdots z_nv_n)$ has the same rightmost $(n-i)$ factors $(z_{i+1}v_{i+1}, \ldots, z_nv_n)$ as w and the same rightmost $(n-i+1)$ *k-records* $z_i, z_{i+1}, \ldots, z_n$ as w.

Consider a biword $\alpha = \binom{u_1u_2\infty}{w_1zu_2}$, where:

(1) u_1, u_2, w_1 are words in the alphabet $X \cup \{*\}$;
(2) u_1 is the non-decreasing rearrangement of w_1;

(3) z is a k-record of the word $w_1 z u_2$.

Such a biword is called a *supercycle*. A supercycle is said to be *initial*, if u_1 and w_1 are empty. The notion of *final* supercycle will be defined shortly.

LEMMA 3. *If α is a supercycle, the factorization*

$$\alpha = \left(\begin{array}{c|cc} u_1 & u_2 & \infty \\ w_1 & z & u_2 \end{array} \right), \tag{9}$$

where u_1 is the non-decreasing rearrangement of some left factor of the bottom word of α and where z is a k-record of the bottom word, is unique. The factorization (9) is called the canonical form of α.

Let α be a supercycle written in its canonical form as in (9) and let $(z_1v_1, z_2v_2, \ldots, z_nv_n)$ be the *k-factorizarion* of zu_2. Then the following factorization of α, indicated by vertical bars,

$$\alpha = \left(\begin{array}{c|cc|cc|c|cc} u_1 & v_1 & z_2 & v_2 & z_3 & \cdots & v_n & \infty \\ w_1 & z_1 & v_1 & z_2 & v_2 & \cdots & z_n & v_n \end{array} \right) \tag{10}$$

is well defined. Call it the *k-factorization* of the supercycle α. The positive integer n, which is the number of factors in the k-factorization of zu_2, is called the *index* of α and denoted by $\mathrm{index}(\alpha)$. A supercycle α is said *final*, if its index is equal to 1.

Let α be a supercycle as shown in (10), supposed to be non final, so that $n \geq 2$. With the notation of (8) the left factor $\binom{u_1 v_1 z_2}{w_1 z_1 v_1}$ of the supercycle α is an element of $\mathcal{D}(u_1v_1, z_1, z_2)$. Apply the inverse transformation T^{-1} to that left factor. We get a biword $\binom{u_1''}{w_1'} \in \mathcal{C}(u_1v_1, z_1, z_2)$. Then form the supercycle

$$\alpha'' = \left(\begin{array}{c|cc|c|cc} u_1'' & v_2 & z_3 & \cdots & v_n & \infty \\ w_1' & z_2 & v_2 & \cdots & z_n & v_n \end{array} \right). \tag{11}$$

Replacing the only occurrence of z_2 in u_1'' by z_1 transforms u_1'' into a true rearrangement u_1' of w_1'. Furthermore, u_1' is non-decreasing and we then obtain a supercycle

$$\alpha' = \left(\begin{array}{c|cc|c|cc} u_1' & v_2 & z_3 & \cdots & v_n & \infty \\ w_1' & z_2 & v_2 & \cdots & z_n & v_n \end{array} \right). \tag{12}$$

Moreover, the above expression derived from the k-factorization of α is precisely the k-factorization of α' and the rightmost k-record of w_1' is equal to z_1. Finally, $\mathrm{index}(\alpha') = n - 1$. Thus the mapping $\tau : \alpha \mapsto \alpha'$ is well defined and satisfies

$$\mathrm{index}(\tau(\alpha)) < \mathrm{index}(\alpha), \tag{13}$$

if α is not final.

LEMMA 4. *Let v a non-decreasing word in the alphabet $X \cup \{*\}$ and let $S(v)$ be the set of the supercycles $\alpha = \binom{u_1 u_2 \infty}{w_1 z u_2}$, whose bottom word $w_1 z u_2$ is a rearrangement of v. If α is not initial, there is a unique $\beta \in S(v)$ such that $\tau(\beta) = \alpha$ and $\mathrm{index}(\alpha) < \mathrm{index}(\beta)$.*

LEMMA 5. *For each supercycle α which is not final, we have*

$$(\mathrm{exc}_k, \mathrm{den}_k)\tau(\alpha) = (\mathrm{exc}_k, \mathrm{den}_k)\alpha.$$

LEMMA 6. *If w is a word in the alphabet X and $\alpha = \binom{u * \infty}{w *}$ is an initial supercycle, then*

$$(\mathrm{exc}_k, \mathrm{den}_k)\alpha = (\mathrm{des}_k, \mathrm{maj}_k)\alpha = (\mathrm{des}_k, \mathrm{maj}_k)w.$$

The bijection of Theorem 2 is constructed as follows:

(1) Let $w = x_1 x_2 \cdots x_m = x_1 u$ be a word in the alphabet $X = [r]$; form the initial supercycle $\alpha = \binom{u*\infty}{w*}$;
(2) Apply the mapping τ to α iteratively until a final supercycle is reached. This makes sense because of (13). Furthermore, when applying τ iteratively, the letter $*$ remains the rightmost letter in all the supercycles within the iteration. Denote by $\tilde{\alpha} = \binom{\overline{w}\infty}{\tilde{w}*}$ the final supercycle obtained. Then $\overline{w}$ is the non-decreasing rearrangement of w and $\tilde{w}$;
(3) Define ρ by $\rho(w) = \tilde{w}$.

Theorem 2 follows from lemma (5) and lemma (6).

Example. Consider the order $1 < 2 < 3 < * < 4 < 5 < \infty$ ($k = 2$ and $4, 5$ large) and start with the word $w = 4,4,5,1,3,1,2,3,5$, so that (indicating k-factorization by vertical bars) the initial supercycle is

$$\alpha = \left(\begin{array}{c|c|c|c} 4\,5\,1 & 3 & 1\,2\,3\,5\,* & \infty \\ 4\,4\,5 & 1 & 3\,1\,2\,3\,5 & * \end{array}\right)$$

First, $T^{-1}\binom{4\,5\,1}{4\,4\,5} = T_1 T_2 \binom{4\,5\,1}{4\,4\,5} = \binom{1\,4\,5}{4\,5\,4}$, so that

$$\alpha'' = \left(\begin{array}{c|c|c|c} 1\,4\,5 & 3 & 1\,2\,3\,5\,* & \infty \\ 4\,5\,4 & 1 & 3\,1\,2\,3\,5 & * \end{array}\right)$$

(in the notation of (11)). To obtain $\alpha' = \tau(\alpha)$ we have to replace $z_2 = 1$ by $z_1 = 4$, so that

$$\alpha_2 = \alpha' = \left(\begin{array}{c|c|c|c} 4\,4\,5 & 3 & 1\,2\,3\,5\,* & \infty \\ 4\,5\,4 & 1 & 3\,1\,2\,3\,5 & * \end{array}\right).$$

Next $T_{-1}\binom{4\,4\,5\,3}{4\,5\,4\,1} = T_1 T_2 T_3 \binom{4\,4\,5\,3}{4\,5\,4\,1} = \binom{3\,4\,4\,5}{4\,5\,1\,4}$ and

$$\alpha'' = \left(\begin{array}{c|c|c} 3\,4\,4\,5 & 1\,2\,3\,5\,* & \infty \\ 4\,5\,1\,4 & 3\,1\,2\,3\,5 & * \end{array}\right).$$

To get the next supercycle we have to replace $z_2 = 3$ by $z_1 = 1$, so that

$$\alpha_3 = \left(\begin{array}{c|c|c} 1\,4\,4\,5 & 1\,2\,3\,5\,* & \infty \\ 4\,5\,1\,4 & 3\,1\,2\,3\,5 & * \end{array}\right).$$

Next the transformation T^{-1} to be applied to $\binom{1\,4\,4\,5\,1\,2\,3\,5\,*}{4\,5\,1\,4\,3\,1\,2\,3\,5}$ is $(T_5T_6T_7T_8)(T_4T_5T_6)(T_3T_4T_5)(T_2T_3T_4)$, as we have to move the second 1, 2, 3 and $*$ to the left. We then get $T^{-1}\binom{1\,4\,4\,5\,1\,2\,3\,5\,*}{4\,5\,1\,4\,3\,1\,2\,3\,5} = \binom{1\,1\,2\,3\,*\,4\,4\,5\,5}{4\,5\,1\,4\,1\,3\,2\,3\,5}$, so that

$$\alpha'' = \left(\begin{array}{c|c} 1\,1\,2\,3\,*\,4\,4\,5\,5 & \infty \\ 4\,5\,1\,4\,1\,3\,2\,3\,5 & * \end{array}\right).$$

Finally, $*$ on the top row is to be replaced by the penultimate k-record, i.e., 3. We get

$$\tilde{\alpha} = \left(\begin{array}{c|c} 1\,1\,2\,3\,3\,4\,4\,5\,5 & \infty \\ 4\,5\,1\,4\,1\,3\,2\,3\,5 & * \end{array}\right).$$

Thus $\rho(w) = 4,5,1,4,1,3,2,3,5$. We can verify that

$$(\mathrm{des}_k, \mathrm{maj}_k)w = (\mathrm{des}_k, \mathrm{maj}_k)\alpha = (\mathrm{exc}_k, \mathrm{den}_k)\alpha = (\mathrm{exc}_k, \mathrm{den}_k)\tilde{\alpha} = (\mathrm{exc}_k, \mathrm{den}_k)\rho(w) = (4, 18).$$

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