Descents in Words

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[summary by Dominique Gouyou-Beauchamps]

1. Introduction

Let S_n denote the symmetric group on $\{1, 2, \ldots, n\}$. For a permutation $\sigma \in S_n$, the *rise set*, *descent set*, *inversion set*, and their cardinalities are respectively defined by

$$\begin{aligned} \operatorname{Ris}\sigma &= \{i : 1 \le i \le n-1, \sigma(i) < \sigma(i+1)\}, & \operatorname{ris}\sigma &= |\operatorname{Ris}\sigma|, \\ \operatorname{Des}\sigma &= \{i : 1 \le i \le n-1, \sigma(i) > \sigma(i+1)\}, & \operatorname{des}\sigma &= |\operatorname{Des}\sigma|, \\ \operatorname{Inv}\sigma &= \{(k,m) : 1 \le k < m \le n, \sigma(m) < \sigma(k)\}, & \operatorname{inv}\sigma &= |\operatorname{Inv}\sigma|. \end{aligned}$$

The *set of common descents* of a pair of permutations $(\sigma_1, \sigma_2) \in S_n^2$ is defined as $\operatorname{DD}(\sigma_1, \sigma_2) = \operatorname{Des}\sigma_1 \cap \operatorname{Des}\sigma_2$ and one notes $\operatorname{dd}(\sigma_1, \sigma_2) = |\operatorname{DD}(\sigma_1, \sigma_2)|$.

Now we recall three definitions of Bessel functions (with $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$):

$$J_\nu(x) = \sum_{n\ge 0} \frac{(-1)^n \left(\frac{1}{2}x\right)^{2n+\nu}}{n!\Gamma(\nu+n+1)}, \qquad \text{(usual)}$$

$$J_\nu(x) = \sum_{n\ge 0} \frac{(-1)^n x^{n+\nu}}{n!(n+\nu)!}, \qquad \text{(combinatorial)}$$

$${}^qJ_\nu(x) = \sum_{n\ge 0} \frac{(-1)^n q^{\binom{n+\nu}{2}} x^{n+\nu}}{(q)_n (q)_{n+\nu}}, \qquad \text{(q-analog).}$$

Perhaps the first combinatorial context for an element of $\{J_\nu\}_{\nu\ge 0}$ was discovered by L. Carlitz, R. Scoville, and T. Vaughan [4].

Theorem 1 ([4, 12]). *The generating function of Bessel type for the sequence of polynomials*

$$a_n(y,q) = \sum_{(\alpha,\beta)\in S_n^2} q^{\operatorname{inv}\alpha} q^{-\operatorname{inv}\beta} y^{\operatorname{dd}(\alpha,\beta)} \qquad \textit{is} \qquad \sum_{n\ge 0} q^{\binom{n}{2}} \frac{a_n(y,q)}{(q)_n(q)_n} x^n = \frac{1-y}{{}^qJ_0(x(1-y)) - y}.$$

Setting $q = 1$ and $y = 0$ in Theorem 1, they deduce that the coefficient $a_n(0,1)$ of $x^n/(n!n!)$ in the series expansion of $1/J_0(x)$ is equal to the number of permutation pairs with no common descents.

First we remark that

$$\sum_{n\ge 0} q^{\binom{n}{2}} \frac{a_n(y,q)}{(q)_n(q)_n} x^n = \frac{1}{1 + \sum_{n\ge 1}(-1)^n (1-y)^{n-1} q^{\binom{n}{2}} x^n/(q)_n^2}. \tag{1}$$

Second we remark that an inversion formula such as (1) involving alternate sums of $(1-y)^{n-1}x^n$ is also present in the well known q-Eulerian polynomials and q-Euler polynomials [2, 3, 16, 17]. More precisely, let $A_n(y,q)$ denote the q-Eulerian polynomials. Then

$$\sum_{n\geq 0} A_n(y,q)\frac{x^n}{(q)_n} = \frac{1}{1+\sum_{n\geq 1}(-1)^n(1-y)^{n-1}x^n/(q)_n}.$$

A natural problem is to give for generating functions of the type

$$F(x,y) = \frac{1}{1+\sum_{n\geq 1}(-1)^n\lambda_n(1-y)^{n-1}x^n}$$

a combinatorial interpretation similar to the one described for the generating function $F(x,0)$ of heaps of pieces [18]. There are a number of powerful theories of inversion [13, 14, 17, 19] for dealing with combinatorial objects having generating functions of type $F(x,0)$. Using two such inversion formulas, we present new derivations of R. P. Stanley's generating functions for generalized q-Eulerian and q-Euler polynomials on r-uples of permutations [17]. We further indicate how one of the inversion formulas gives V. Diekert's lifting to the free monoïd of an inversion theorem of P. Cartier and D. Foata [5, 7]. The inversion theorems we use enumerate words in the free monoïd by adjacencies.

2. From the free to the trace monoïd

Let X be an alphabet. The empty word will be denoted by 1. The set of all words formed with letters in X by means of the concatenation product is known as the *free monoïd* generated by X and is denoted by X^*. In $\mathbb{Z}\langle\langle X\rangle\rangle$, the ring of formal power series of words in X^* with integer coefficients, the following inversion formula holds: $X^* = 1/(1-X)$.

Let θ be an irreflexive symmetric binary relation on X. Define $\equiv_\theta$ to be the binary relation (induced by θ) on X^* consisting of the set of pairs (w,v) of words such that there is a sequence $w = w_0, w_1, \ldots, w_m = v$ where each w_i is obtained by transposing a pair of letters in w_{i-1} that are consecutive and contained in θ.

Clearly, $\equiv_\theta$ is an equivalence relation on X^*. The quotient of X^* by $\equiv_\theta$ gives the *partially commutative monoïd* (or *trace monoïd*) induced by θ and denoted by $M(X,\theta)$. The equivalence class $\hat{w}$ of $w \in X^*$ is referred to as the *trace* of w.

A word $w = x_1x_2\cdots x_n \in X^*$ is said to be a *basic monomial* if $x_i\theta x_j$ for all $i \neq j$. Note that all the letters of a basic monomial are distinct. A trace $\hat{w}$ is said to be *θ-trivial* if any one of its representatives is a basic monomial. Letting $\mathcal{T}^+(X,\theta)$ be the set of θ-trivial traces, the inversion formula of Möbius type reads as follows.

THEOREM 2 (P. CARTIER AND D. FOATA). *For θ an irreflexive symmetric binary relation on X, the traces in $M(X,\theta)$ are generated by*

$$\sum_{\hat{w}\in M(X,\theta)} \hat{w} = \frac{1}{1+\sum_{\hat{t}\in\mathcal{T}^+(X,\theta)}(-1)^{l(\hat{t})}\hat{t}},$$

where $l(\hat{t})$ denotes the length of any representative of $\hat{t}$.

In terms of heaps of pieces, the Cartier-Foata's theorem is nothing but the inversion lemma for heap monoïd [18, prop. 5.1].

A natural question to ask is whether $\hat{w}$ and $\hat{t}$ can be replaced by some canonical representatives so that Theorem 2 remains true as a formula in the free monoïd X^*. As resolved by V. Diekert [6, 7], such canonical representatives exist if and only if θ admits a transitive orientation.

To be precise, a subset $\vec{\theta}$ of θ is an *orientation* of θ if θ is a disjoint union of $\vec{\theta}$ and $\{(x,y) : (y,x) \in \vec{\theta}\}$. The set of $t = t_1 t_2 \cdots t_n \in X^*$ satisfying $t_1 \vec{\theta} t_2 \vec{\theta} \cdots \vec{\theta} t_n$ is denoted by $T^+(X, \vec{\theta})$. Note that $T^+(X,\vec{\theta})$ is a set of representatives for the θ-trivial traces $\mathcal{T}^+(X,\theta)$ whenever $\vec{\theta}$ is transitive. A word $w = x_1 x_2 \cdots x_n \in X^*$ is said to have a $\vec{\theta}$-*adjacency* in position k if $x_k \vec{\theta} x_{k+1}$. We denote the number of $\vec{\theta}$-adjacencies of w by $\vec{\theta}\,\mathrm{adj}\, w$. Although V. Diekert did not explicitly introduce the notion of a $\vec{\theta}$-adjacency, his lifting theorem may be paraphrased as follows.

Theorem 3 (V. Diekert). *Let θ be an irreflexive symmetric binary relation on X and $\vec{\theta}$ be an orientation of θ. Then, $\vec{\theta}$ is transitive if and only if there exists a complete set W of representatives for the traces of $M(X,\theta)$ such that*

$$\sum_{w \in W} w = \frac{1}{1 + \sum_{t \in T^+(X,\vec{\theta})} (-1)^{l(t)} t}.$$

Moreover, $W = \{w \in X^* : \vec{\theta}\,\mathrm{adj}\, w = 0\}$.

3. Descents in a word

Now X is a totally ordered alphabet. We say that a word $w = x_1 \cdots x_i x_{i+1} \cdots x_n$ of X^* has a θ-*descent* in position i when $x_i \theta x_{i+1}$ and $x_i > x_{i+1}$. We note $x \gg_\theta y$ (resp. $x \ll_\theta y$) when $x \theta y$ and $x > y$ (resp. $x < y$). Let $I^+ = \{x_1 \cdots x_n \in X^*, n > 0, x_1 \gg_\theta x_2 \gg_\theta \cdots \gg_\theta x_{n-1} \gg_\theta x_n\}$. Let $w \in X^*$, we denote by $\theta\,\mathrm{des}(w)$ the number of its θ-descents.

Theorem 4 ([9]). *The following equality holds in the free monoïd*

$$\sum_{w \in X^*} y^{\theta\,\mathrm{des}(w)} w = \frac{1}{1 - \sum_{t \in I^+} (y-1)^{|t|-1} t}. \tag{2}$$

When $\gg_\theta$ is transitive, setting $y = 0$ in (2) gives the lifting of Theorem 2 to the free monoïd as stated in Diekert's Theorem 3. We close this section with two examples.

Example (Transitive case). Let $X = \{a,b,c\}, a < b < c$ with $\theta = \{(a,b),(b,a),(a,c),(c,a)\}$. The θ-descents of a word correspond to factors ba and ca. Then $\gg_\theta$ is a transitive relation. Note that $I^+ = \{a,b,c,ba,ca\}$ is a complete set of the representatives for the θ-trivial traces $\mathcal{T}^+(X,\theta)$. From (2), we have

$$\sum_{w \in X^*} y^{\theta\,\mathrm{des}(w)} w = \frac{1}{1 - (a+b+c) + (1-y)(ba+ca)}.$$

Setting $y = 0$ gives an identity that can be viewed as having been lifted from the trace monoïd as in Theorem 3.

Example (Non-transitive case). With the same alphabet, let $\theta = \{(a,b),(b,a),(b,c),(c,b)\}$. The θ-descents of a word correspond to factors ba and cb. Then $\gg_\theta$ is not a transitive relation. Observe that the word cba in $I^+ = \{a,b,c,ba,cb,cba\}$ is not a θ-trivial trace. Also, the class of cba is $\{cba, cab, bca\}$ and contains two words having no θ-descents (or no $\gg_\theta$-adjacencies). Nevertheless, (2) implies

$$\sum_{w \in X^*} y^{\theta\,\mathrm{des}(w)} w = \frac{1}{1 - (a+b+c) + (1-y)(ba+cb) - (1-y)^2 cba}.$$

4. Adjacencies in words

Let X be an alphabet. From X, we construct the *adjacency* alphabet $A = \{a_{xy} : (x,y) \in X \times X\}$. The *adjacency monomial* and the *sieve polynomial* for $w = x_1 x_2 \cdots x_n \in X^*$ of length $n \geq 2$ are defined respectively as $a(w) = a_{x_1x_2} a_{x_2x_3} \cdots a_{x_{n-1}x_n}$ and $\overline{a}(w) = (a_{x_1x_2} - 1)(a_{x_2x_3} - 1) \cdots (a_{x_{n-1}x_n} - 1)$. For $0 \leq n \leq 1$, we set $a(w) = \overline{a}(w) = 1$. In $\mathbb{Z}[A]\langle\langle X \rangle\rangle$, the algebra of formal power series of words in X^* with polynomial coefficients, the following inversion formula holds:

Theorem 5 ([10, 14, 17, 19]). *According to the adjacencies, the words in X^* are generated by*

$$\sum_{w \in X^*} a(w) w = \frac{1}{1 - \sum_{w \in X^+} \overline{a}(w) w}. \tag{3}$$

If for $u, v \in X$ we set $a_{uv} = y$ when $x \gg_\theta y$ and $a_{uv} = 1$ otherwise, Theorem 2 can be seen as a corollary of Theorem 5. In passing, we mention that J. Hutchinson and H. Wilf [15] have given a closed formula for counting words by adjacencies.

Example. The applications we give rely on the fact that setting $a_{xy} = 1$ eliminates all words containing xy as a factor from the right-hand side of (3). Suppose that $X = \{a, b, c\}$. Setting $a_{aa} = r$, $a_{ab} = s$, $a_{ac} = t$ and the remaining $a_{ij} = 1$ in Theorem 5 yields

$$\sum_{w \in X^*} a(w) w = \frac{1}{1 - \sum_{w \in B} \overline{a}(w) w},$$

where $B = \{a^n \mid n \geq 1\} \cup \{a^n b \mid n \geq 0\} \cup \{a^n c \mid n \geq 0\}$. Thus

$$\begin{aligned}
&\sum_{w \in X^*} a(w) w \\
&\quad = \left[1 - \sum_{n \geq 1} (r-1)^{n-1} a^n - b - \sum_{n \geq 1} (r-1)^{n-1}(s-1) a^n b - c - \sum_{n \geq 1} (r-1)^{n-1}(t-1) a^n c\right]^{-1} \\
&\quad = \frac{1 + a - ra}{1 - ra - b - c + (r-s)ab + (r-t)ac}.
\end{aligned}$$

5. The insertion-shift bijection

In applying Theorem 5 to the enumeration of permutations, we make repeated use of the insertion-shift bijection [8] that associates a finite sequence of non-negative integers to a pair (σ, λ) where σ is a permutation and λ is a partition.

Let $\mathbb{N}_+^n$ be the set of words of length n in $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$. The *rise set*, *rise number*, *inversion number*, and *norm* of $w = i_1 i_2 \cdots i_n \in \mathbb{N}_+^n$ are respectively defined by

$$\begin{aligned}
&\operatorname{Ris} w = \{k : 1 \leq k \leq n-1, i_k < i_{k+1}\}, && \operatorname{ris} w = |\operatorname{Ris} w|, \\
&\operatorname{inv} w = |\{(k,m) : 1 \leq k < m \leq n, i_k > i_m\}|, && \|w\| = i_1 + i_2 \cdots + i_n.
\end{aligned}$$

The set of non-decreasing words in $\mathbb{N}_+^n$ (i.e., partitions with at most n parts) will be denoted by P_n. We have to construct inductively a bijection $f_n : \mathbb{N}_+^n \to S_n \times P_n$. If $n = 1$, then the map $i \mapsto (1, i)$ does the job. Suppose f_{n-1} exists and let $w = i_1 i_2 \cdots i_n \in \mathbb{N}_+^n$. Applying f_{n-1} to the first $n-1$ letters of w gives a pair (α, δ) in $S_{n-1} \times P_{n-1}$. Note $\delta_0 = 0$, $\delta_n = \infty$ and $\delta = (\delta_1 \cdots \delta_{n-1})$. Determining k such that $\delta_{k-1} \leq i_n < \delta_k$, we define $f_n(w)$ to be the pair

$$\Big(\alpha(1) \cdots \alpha(k-1)\, n\, \alpha(k) \cdots \alpha(n-1), \delta_1 \cdots \delta_{k-1}\, i_n\, (\delta_k - 1) \cdots (\delta_{n-1} - 1)\Big).$$

LEMMA 1 ([8]). *For $n \geq 1$ and for $w \in \mathbb{N}_+^n$, if w is mapped to the pair (σ, λ) by the bijection $f_n : \mathbb{N}_+^n \to S_n \times P_n$, then* $\operatorname{Ris} w = \operatorname{Ris} \sigma^{-1}$ *and* $\|w\| = \operatorname{inv} \sigma + \|\lambda\|$.

EXAMPLE. The word $w = 372314 \in \mathbb{N}_+^6$ is mapped by f_6 to the pair $(\sigma, \lambda) = (531426, 111244) \in S_6 \times P_6$. Noting that $\sigma^{-1} = 352416$, we see that $\operatorname{Ris} w = \{1, 3, 5\} = \operatorname{Ris} \sigma^{-1}$ and that $\|w\| = 20 = \operatorname{inv} \sigma + \|\lambda\| = 7 + 13$.

6. q-Eulerian polynomials and bibasic Bessel functions

As the first application of Theorem 1, we derive a generating function for the sequence

$$A_n(t, q) = \sum_{\sigma \in S_n} t^{\operatorname{ris} \sigma} q^{\operatorname{inv} \sigma}.$$

The polynomial $A_n(t, 1)$ is the n-th *Eulerian polynomial.*

We set $a_{ij} = t$ if $i \leq j$ and $a_{ij} = 1$ otherwise. Theorem 1 reduces to

$$\sum_{w \in \mathbb{N}_+^*} t^{\operatorname{ris} w} w = \frac{1}{1 - \sum_{n \geq 1} (t-1)^{n-1} \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} i_1 i_2 \cdots i_n}. \tag{4}$$

Using (4) and lemma 1, we have the following form for the generating function:

$$\sum_{n \geq 0} \frac{A_n(t, q) z^n}{(q)_n} = \frac{1}{1 - \sum_{n \geq 1} (t-1)^{n-1} z^n / (q)_n} = \frac{1-t}{E\left(-z(1-t), q\right) - t}, \tag{5}$$

where $E(z, q) = \sum_{n \geq 0} z^n / (q)_n$ is a well-known q-analog of e^z.

Now we derive a generating function for the sequence

$$B_n(t, q_1, q_2) = \sum_{(\sigma_1, \sigma_2) \in S_n^2} t^{\operatorname{dd}(\sigma_1, \sigma_2)} q_1^{\operatorname{inv} \sigma_1} q_2^{\operatorname{inv} \sigma_2}.$$

We use the alphabet $X' = \{\binom{a}{a'}, (a, a') \in \mathbb{N}_+^2\}$ and for letters $\mathbf{i} = (i_1, i_2)$ and $\mathbf{j} = (j_1, j_2)$, we set $a_{\mathbf{ij}} = t$ if $i_1 \leq i_2$ and $j_1 \leq j_2$, and $a_{\mathbf{ij}} = 1$ otherwise. Repeating (5) with appropriate modifications gives

$$\sum_{n \geq 0} \frac{B_n(t, q_1, q_2) z^n}{(q_1)_n (q_2)_n} = \frac{1}{1 - \sum_{n \geq 1} (t-1)^{n-1} \frac{z^n}{(q_1)_n (q_2)_n}} = \frac{1-t}{J\left(-z(1-t), q_1, q_2\right) - t},$$

where $J(z, q) = \sum_{n \geq 0} (-1)^n z^n / (q_1)_n (q_2)_n$ is a bibasic Bessel function.

7. q-Euler polynomials

D. André [1] showed that if E_n denotes the number of up-down alternating permutations in S_n (that is, $\sigma \in S_n$ that $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \cdots$), then

$$\sum_{n \geq 0} E_n \frac{z^n}{n!} = \frac{1 + \sin z}{\cos z}.$$

The number E_n is known as the n-th *Euler number.*

We now apply Theorem 5 to the more general problem of counting the set of *odd-up permutations* $\mathcal{O}_n = \{\sigma \in S_n : \sigma(1) < \sigma(2), \sigma(3) < \sigma(4) \cdots\}$ by inversion number and by the *number of even indexed rises* $\operatorname{ris}_2 \sigma = |\{k \in \operatorname{Ris} \sigma : k \text{ is even}\}|$. We begin by determining a generating function for

$$C_{2n}(t, q) = \sum_{\sigma \in \mathcal{O}_{2n}} t^{\operatorname{ris}_2 \sigma} q^{\operatorname{inv} \sigma}.$$

Note that $C_{2n}(0, 1) = E_{2n}$.

Let $X = \{\mathbf{i} = (i_1, i_2) : i_1, i_2 \in \mathbb{N}_+ \text{ with } i_1 \le i_2\}$. For letters $\mathbf{i} = (i_1, i_2)$ and $\mathbf{j} = (j_1, j_2)$, we set $a_{\mathbf{ij}} = t$ if $i_2 \le j_1$ and $a_{\mathbf{ij}} = 1$ otherwise. Then we have

$$\sum_{n \ge 0} C_{2n}(t, q) \frac{z^{2n}}{(q)_{2n}} = \frac{1}{1 - \sum_{n \ge 1} (t-1)^{n-1} \frac{z^{2n}}{(q)_{2n}}}.$$

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