Summation of series solutions of linear differential equations

Michèle Loday-Richaud
Université d'Orsay

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[summary by Bruno Salvy]

Introduction

It is very rare that a linear differential equation with analytic coefficients admits a closed-form solution. However, a lot of local informations can be obtained directly from the equation. For instance, it is well known that at an ordinary point the Taylor series of a solution can be computed by undeterminate coefficients. It is also possible to compute formal local expansions of the solutions of a linear differential equation, or equivalently a first order linear system, in the neighbourhood of a singularity.

Singularities are found to be either poles of the coefficients or roots of the coefficient of highest order. Suppose the singularity of the equation is at the origin, or you move it there by a change of variable. Then a fundamental object to compute is the *Newton polygon*. It is obtained by taking the upper-left convex hull of the set of points $(i, j-i)$ such that $x^j\partial^i$ appears with a non-zero coefficient in the Taylor series of the differential equation. On the Newton polygon one can read whether the singular point is *regular* or *irregular*. In the former case the polygon has only one edge which is horizontal, while it has several ones in the latter case. This distinction corresponds to very different formal solutions.

When the point is ordinary the power series obtained by substitution form a basis of *convergent* series solutions. When the point is a regular singular point, one can obtain a basis of formal solutions of the form

$$\sum f_{i,j}\, x^{\lambda_i} \log^j x,$$

where the $f_{i,j}$ are convergent power series and the λ_i are algebraic numbers. When the singularity is irregular, the formal solutions look like

$$\sum \hat{f}_{i,j,k} x^{\lambda_i} \log^j x\, e^{q_k(1/x)},$$

where the $\hat{f}_{i,j,k}$ are formal power series that are usually divergent, and the q_k are polynomials in some rational power of x.

The question naturally arises of "computing" actual solutions from these series. In the ordinary and regular singular cases, there is no theoretical difficulty since the series converge in the neighbourhood of the origin. In the divergent case the Main Asymptotic Existence Theorem asserts that each formal expansion is the asymptotic expansion of an actual solution in some angular sector of sufficiently small amplitude. Unfortunately, in a small sector there are also *flat* analytic functions whose power series expansion at the origin is 0. The problem of summability is to find a good way to associate a distinguished function to a formal solution while preserving good properties. In the process, one finds explicit formulæ that permit numerical computations of the solutions.

1. The Laplace-Borel method

Given a divergent formal power series $\hat{f} = \sum a_n x^{n+1}$, the Laplace-Borel method in a direction d consists in first applying a *Borel transform* to $\hat{f}$, transforming it into $\phi(\xi) = \sum \frac{a_n}{\Gamma(n+1)}\xi^n$, continuing ϕ analytically along d, and then applying a *Laplace transform* to ϕ, yielding

$$f(x) = \int_d \phi(\xi) e^{-\xi/x}\, d\xi.$$

When this method works, the resulting function f is an actual solution of the linear differential equation of which $\hat{f}$ was a formal solution. From this expression it is possible to compute numerical values (see [11]). When a series is Laplace-Borel summable for almost all directions d, it is said to be Laplace-Borel summable.

For this method to work, it is necessary that ϕ be a convergent series and that its analytic continuation on a sector containing the direction d have an exponential growth of order at most 1 at infinity. A sufficient condition for this to work is that the series is a solution of a differential equation whose Newton polygon has only one oblique edge, with slope 1.

Unfortunately this method does not apply to all the formal solutions of linear differential equations and has to be generalized to k-summability and multisummability.

2. k-summability

A formal power series $\sum a_n x^n$ is defined to be *k-Gevrey* when $\sum a_n x^n / \Gamma(1 + n/k)$ is convergent. A holomorphic function f on an open set V with vertex 0 is said to be *asymptotically k-Gevrey* on V if there exist a formal power series $\hat{f} = \sum a_n x^n$, and two real numbers A and K such that for any positive integer N the following inequality holds uniformly in V

$$|f(x) - \sum_{n<N} a_n x^n| \leq K\Gamma(1 + \frac{N}{k}) A^n |x|^n.$$

In this case $\hat{f}$ is necessarily k-Gevrey.

A formal power series $\hat{f}(x)$ is said to be *k-summable* in direction d when $\hat{g}(t) = \hat{f}(t^{1/k})$ is Borel-Laplace summable in direction d. The method to resum these series is to substitute $t^{1/k}$ by x, then apply the Laplace-Borel method (extended so as to accomodate ramified series), and then replace back $t^{1/k}$ by x. While doing this it is necessary to be careful about directions of integration and amplitude of domains. This methods applies when the series is solution of a linear differential equation whose Newton polygon has only one oblique edge, with slope k.

3. Multisummability

Not all formal series solutions of linear differential equations are k-summable for some k. For instance if $k_1 \neq k_2$, and f_1 and f_2 are respectively non-convergent k_1 and k_2-summable series in direction d, then $f_1 + f_2$ is not k-summable in direction d for any value of k. The main theorem [2, 9] is that all the elements of the differential algebra generated by f_1 and f_2 can be written as a sum of a k_1 and a k_2-summable series. This extends to algebras generated by more than two series. Such series are called *multisummable* of levels $(k_1, k_2, \ldots)$ in direction d. Although the representation of a divergent series as an element of this differential algebra is not unique, it can be shown that by resumming each of the series by its k-Borel-Laplace method in direction d and summing the results yields a result which depends only on the initial series and on d. Besides, the values $k_1, k_2, \ldots$ can all be deduced from the slopes of the Newton polygon of an equation having the series as solution.

The next problem is that this decomposition is not given by the above theorems. To actually compute the sum, two methods are known.

3.1. Ecalle's acceleration method. Suppose that $\hat{f} = \hat{f}_1 + \hat{f}_2$, with f_1 k_1-summable and f_2 k_2-summable in direction d. It is impossible to apply successively k_1 and k_2 summability in direction d, because the asymptotic growth of the k_1-Borel transform of $\hat{f}$ is too fast. Instead of this, the idea [5, 6, 7] is to compute an operator which performs simultaneously the k_1-Laplace transform and the k_2-Borel transform ($k_1 < k_2$). This operator is called *Ecalle accelerator*. Its value is

$$\mathbf{A}_{(k_1,k_2)}(\phi)(\xi_2) = \int_d \frac{1}{\xi_2^{k_2}} \phi(u)\, \mathcal{C}_{k_2/k_1}\big[\big(\frac{u}{\xi_2}\big)^{k_1}\big]\, du, \quad \text{where} \quad \mathcal{C}_a(\xi) = \frac{1}{2i\pi} \int_{\mathcal{H}} e^{v - \xi v^{1/a}}\, dv,$$

and $\mathcal{H}$ is a Hankel contour. Apart from the case $a = 2$ these are new special functions.

This method applies to any finite number of summands.

3.2. Balser's method. This method [3] first computes the κ_1 and κ_2-Borel transforms, and then all the corresponding κ-Laplace transforms in reverse order. The values of the κ's are related to the previous values of k by

$$\frac{1}{\kappa_p} = \frac{1}{k_p}, \qquad \frac{1}{\kappa_j} = \frac{1}{k_j} - \frac{1}{k_{j+1}} \quad (j < p).$$

Here also, the sum depends only on the initial formal power series, and it happens to be the same as given by Ecalle's method.

Final comments

Most of the proofs of these theorems involve highly sophisticated tools in algebra and analysis. The reader who wants to see more of it should consult the references.

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