Enumerations related to automorphisms of rooted tree structures

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Abstract

The goal of this paper is to present a panorama of the fundamental properties of *cycle index series* and *asymmetry index series* within enumerative combinatorics, as well as a few concrete applications. A given structure is said to be *asymmetric* if its automorphism group reduces to the identity. We introduce an *asymmetry indicator series* $\Gamma_F(x_1, x_2, x_3, \ldots)$ by means of which we study the correspondence $F \to \bar{F}$ in connection with the various operations existing in the theory of species of structures. It is shown that all these operations are automatically computable but this aspect is not developed in the summary.

1. Species and Asymmetry Index Series

Given any finite set U, let us denote by $A[U]$ the set of all *rooted trees* having U as underlying set of vertices. Clearly, every bijection $\beta : U \to V$ between finite sets induces another bijection which we denote by $A[\beta] : A[U] \to A[V]$ and call the *transportation* of rooted trees along β (we replace each vertex u in a by the corresponding vertex $\beta(u)$). Of course, transportation commutes with composition in the following way: given any two successive bijections $\beta : U \to V, \beta' : V \to W$, we have $A[\beta' \circ \beta] = A[\beta'] \circ A[\beta]$ and $A[1_U] = 1_{A[U]}$ (where 1_U denotes, as usual, the identity bijection of a finite set U into itself).

A *combinatorial species* [5] is a functor from the category of finite sets and bijections into itself. In other words, a combinatorial species is a rule F that associates a finite set $F[U]$ to any finite set U and a bijection $F[\beta] : F[U] \to F[V]$ to any bijection $\beta : U \to V$. An element $s \in F[U]$ is called an *F-structure* on the underlying set U. The bijection $F[\beta]$ is called the *transportation* of F-structures *along* β.

In the case of *weighted species* F, each F-structure s is given a weight $w_F(s)$ in a certain commutative ring $\mathcal{R}$ and the transportation $F[\beta]$ must preserve these weights.

Given a species F and two F-structures $s \in F[U]$ and $s' \in F[V]$, an *isomorphism* β from s to s' is a bijection $\beta : U \to V$ such that $F[\beta](s) = s'$. Two isomorphic F-structures are said to be of the same *type*. An *automorphism* of s is an isomorphism from s to s. The automorphisms of any given F-structure s form a group called the *automorphism group* of s. When this group is trivial, the structure s is said to be *asymmetric*.

For each integer n, consider now the set $\underline{n} = \{1, 2, \ldots, n\}$. It is easy to see that any species F induces, by transportation, a countable family of actions of the symmetric group S_n:

$$S_n \times F[\underline{n}] \to F[\underline{n}], \quad n = 0, 1, 2, \ldots.$$

Given a weighted species F, the formal power series

$$F(x)=\sum_{n\geq 0} f_n \frac{x^n}{n!}, \quad \tilde{F}(x)=\sum_{n\geq 0} \tilde{f}_n x^n, \quad \bar{F}(x)=\sum_{n\geq 0} \bar{f}_n x^n,$$

whose coefficients are defined by

f_n = the sum of the weights of the F-structures on any n-element set
= the sum of the weights of the elements of $F[\underline{n}]$,

$\tilde{f}_n$ = the sum of the weights of the types of F-structures on any n-element set
= the sum of the weights of the orbits of the action $S_n \times F[\underline{n}] \to F[\underline{n}]$,

$\bar{f}_n$ = the sum of the weights of the types of asymmetric F-structures on any n-element set
= the sum of the weights of the $n!$-point orbits of the action $S_n \times F[\underline{n}] \to F[\underline{n}]$,

are respectively called the (exponential) *generating series* of F, the *types generating series* of F, and the *asymmetry types generating series* of F.

EXAMPLE. For every $n \geq 0$ let

a_n = the number of rooted trees on n given vertices
= the number of elements of $A[\underline{n}]$,

$\tilde{a}_n$ = the number of types of of rooted trees on n vertices
= the number of orbits of the action $S_n \times A[\underline{n}] \to A[\underline{n}]$,

$\bar{a}_n$ = the number of types of asymmetric rooted trees on n vertices
= the number of $n!$-point orbits of the action $S_n \times A[\underline{n}] \to A[\underline{n}]$.

These sequences of numbers can be "encoded" into the series

$$A(x)=\sum_{n\geq 0} a_n \frac{x^n}{n!} = x+2\frac{x^2}{2!}+9\frac{x^3}{3!}+64\frac{x^4}{4!}+625\frac{x^5}{5!}+7776\frac{x^6}{6!}+117649\frac{x^7}{7!}+\cdots,$$

$$\tilde{A}(x)=\sum_{n\geq 0} \tilde{a}_n x^n = x+x^2+2x^3+4x^4+9x^5+20x^6+48x^7+115x^8+\cdots,$$

$$\bar{A}(x)=\sum_{n\geq 0} \bar{a}_n x^n = x+x^2+x^3+2x^4+3x^5+6x^6+12x^7+25x^8+\cdots.$$

Given two species F and G, other species can be constructed: the *sum* $F+G$, the *product* $F\cdot G$, the *substitution* $F(G)$ (also denoted $F\circ G$), and the *derivative* F' (also denoted dF/dX). The generic structures belonging to each of these species are described as follows:

(1) s is an $(F+G)$-structure on U *iff* s is an F-structure on U or a G-structure on U (the "or" is an "exclusive or"),
(2) s is an $(F\cdot G)$-structure on U *iff* $s=(f,g)$ where f is an F-structure on U_1, g is a G-structure on U_2, and $U_1\cup U_2=U$, $U_1\cap U_2=\emptyset$,
(3) s is an $F(G)$-structure on U *iff* $s=(f,\gamma)$ where γ is a set of G-structures having disjoint underlying sets whose union is U, and f is an F-structure on the set γ (the assumption $G[\emptyset]=\emptyset$ is made in order to have a finite number of $F(G)$-structures on each U),
(4) s is an F'-structure on U *iff* s is an F-structure on the augmented set $U\cup\{\star\}$, where $\star$ denotes a point outside U.

The passage from species to series satisfies the following properties:

– The transformation $F \to F(x)$ commutes with combinatorial sums, products, substitutions, and derivations:

$$(F+G)(x) = F(x) + G(x), \qquad (F{\cdot}G)(x) = F(x){\cdot}G(x),$$
$$(F\circ G)(x) = F(G(x)), \qquad F'(x) = dF(x)/dx.$$

– The transformations $F \to \tilde{F}(x)$ and $F \to \bar{F}(x)$ commute with combinatorial sums and products but do not commute, in general, with substitutions and derivations:

$$\widetilde{(F+G)}(x) = \tilde{F}(x) + \tilde{G}(x), \qquad \overline{(F+G)}(x) = \bar{F}(x) + \bar{G}(x),$$
$$\widetilde{(F{\cdot}G)}(x) = \tilde{F}(x){\cdot}\tilde{G}(x), \qquad \overline{(F{\cdot}G)}(x) = \bar{F}(x){\cdot}\bar{G}(x),$$
$$\widetilde{(F\circ G)}(x) \neq \tilde{F}(\tilde{G}(x)), \qquad \overline{(F\circ G)}(x) \neq \bar{F}(\bar{G}(x)),$$
$$\widetilde{F'}(x) \neq d\tilde{F}(x)/dx, \qquad \overline{F'}(x) \neq d\bar{F}(x)/dx.$$

Consider an infinite sequence $t = (t_1, t_2, t_3, \ldots)$ of distinct formal "weights" and, given a finite set U, define an F_t-structure on U as being a couple $s = (f, v)$ where f is an F-structure on U and $v : U \to \{1, 2, 3, \ldots\}$ is a function that assigns an arbitrary positive integer to each element of U. Define the t-weight of the structure s by $w(s) = \prod_{u\in U} t_{v(u)}$.

Given a bijection $\beta : U \to V$, define the transportation $F_t[\beta] : F_t[U] \to F_t[V]$ by

$$F_t[\beta](s) = (F[\beta](f), v\circ\beta^{-1}).$$

Of course, the two series $\tilde{F}_t(x)$ and $\bar{F}_t(x)$ can be associated to the weighted species F_t and each series is easily seen to be a symmetric function of the t_i's [11].

Let F be any species and $t = (t_1, t_2, t_3, \ldots)$ be a countable sequence of formal variables related to the variables $x_1, x_2, x_3, \ldots$ by the equations

$$x_k = t_1^k + t_2^k + t_3^k + \cdots, \quad (k\text{-th power sum}), \quad k = 1, 2, 3, \ldots.$$

The *cycle index series* Z_F and the *asymmetry index series* Γ_F are defined by

$$Z_F(x_1, x_2, x_3, \ldots) = \text{the expression of the symmetric function } \tilde{F}_t(x)\,|_{x:=1} \text{ of } t_1, t_2, t_3, \ldots \text{ in terms of the variable } x_1, x_2, x_3, \ldots,$$
$$Z_\Gamma(x_1, x_2, x_3, \ldots) = \text{the expression of the symmetric function } \bar{F}_t(x)\,|_{x:=1} \text{ of } t_1, t_2, t_3, \ldots \text{ in terms of the variable } x_1, x_2, x_3, \ldots.$$

It turns out that the cycle index series Z_F is the sum, over n, of the classical Pólya's cycle index polynomials of the family of actions $S_n \times F[\underline{n}] \to F[\underline{n}]$, of the symmetric group S_n, $n \geq 0$. Examples show that Γ_F contains informations independent of Z_F (and vice versa). Using the theory of symmetric functions and collecting monomials in $x_1, x_2, x_3, \ldots$, both series can be written in the "standard form"

$$f(x_1, x_2, x_3, \ldots) = \sum_{n\geq 0}\sum_{\sigma\vdash n} f_\sigma \frac{x_1^{\sigma_1} x_2^{\sigma_2}\cdots x_n^{\sigma_n}}{1^{\sigma_1}\sigma_1! 2^{\sigma_2}\sigma_2!\cdots n^{\sigma_n}\sigma_n!},$$

where the coefficients f_σ satisfy

$$f_\sigma \in \mathbb{N} \text{ if } f = Z_F, \qquad \text{while} \qquad f_\sigma \in \mathbb{Z} \text{ if } f = \Gamma_F.$$

Species	**F**	F	$\tilde{F}$	$\bar{F}$	Z_F	Γ_F
singleton	X	x	x	x	x_1	x_1
pair	E_2	$\frac{x^2}{2!}$	x^2	0	$\frac{1}{2}(x_1^2 + x^2)$	$\frac{1}{2}(x_1^2 - x^2)$
set	E	$\exp(x)$	$\frac{1}{1-x}$	$1+x$	$\exp\left(\sum_{n\geq 1} \frac{x_n}{n}\right)$	$\exp\left(\sum_{n\geq 1}(-1)^{n-1}\frac{x_n}{n}\right)$
subset	$\mathcal{P}$	$\exp(2x)$	$\frac{1}{(1-x)^2}$	$(1+x)^2$	$\exp\left(2\sum_{n\geq 1}(-1)^{n-1}\frac{x_n}{n}\right)$	$\exp\left(2\sum_{n\geq 1}(-1)^{n-1}\frac{x_n}{n}\right)$
list	L	$\frac{1}{1-x}$	$\frac{1}{1-x}$	$\frac{1}{1-x}$	$\frac{1}{1-x_1}$	$\frac{1}{1-x_1}$
cycle	C	$\ln(\frac{1}{1-x})$	$\frac{x}{1-x}$	x	$\sum_{n\geq 1}\frac{\phi(n)}{n}\ln\left(\frac{1}{1-x_n}\right)$	$\sum_{n\geq 1}\frac{\mu(n)}{n}\ln\left(\frac{1}{1-x_n}\right)$
permutation	S	$\frac{1}{1-x}$	$\prod_{n\geq 1}\frac{1}{1-x^n}$	$1+x$	$\prod_{n\geq 1}\frac{1}{1-x_n}$	$\frac{1-x_2}{1-x_1}$

TABLE 1. Basic species and their generating series. Here $\phi(n)$ and $\mu(n)$ respectively denote the classical Euler and Möbius functions of n.

The notation $\sigma \vdash n$ means that $\sigma = (\sigma^1, \sigma^2, \ldots, \sigma^n)$ runs through the partitions of n, and σ_i is the number of parts of size i in σ.

The transformations $F \to Z_F$ and $F \to \Gamma_F$ both commute with combinatorial sums, products, substitutions and derivations:

$$Z_{F+G} = Z_F + Z_G, \qquad Z_{F\cdot G} = Z_F \cdot Z_G, \qquad Z_{F\circ G} = Z_F \circ Z_G, \qquad Z_{F'} = \frac{\partial Z_F}{\partial x_1}, \tag{1}$$

$$\Gamma_{F+G} = \Gamma_F + \Gamma_G, \qquad \Gamma_{F\cdot G} = \Gamma_F \cdot \Gamma_G, \qquad \Gamma_{F\circ G} = \Gamma_F \circ \Gamma_G, \qquad \Gamma_{F'} = \frac{\partial \Gamma_F}{\partial x_1}, \tag{2}$$

where $Z_F \circ Z_G$ (resp. $\Gamma_F \circ \Gamma_G$) denotes the plethystic substitution of the series Z_F and Z_G (resp. Γ_F and Γ_G). The *plethysm* $Z_F \circ Z_G$ of two series $Z_F = f(x_1, x_2, x_3, \ldots)$ and $Z_G = g(x_1, x_2, x_3, \ldots)$ is the series $f(g_1, g_2, g_3, \ldots)$, where $g_k(x_1, x_2, x_3, \ldots) = g(x_k, x_{2k}, x_{3k}, \ldots)$ [2].

The series $F(x)$, $\tilde{F}(x)$, and $\bar{F}(x)$ can be computed from Z_F and Γ_F by making use of the following remarkable formulas:

$$F(x) = Z_F(x, 0, 0, \ldots) = \Gamma_F(x, 0, 0, \ldots), \tag{3}$$

$$\tilde{F}(x) = Z_F(x, x^2, x^3, \ldots), \qquad \bar{F}(x) = \Gamma_F(x, x^2, x^3, \ldots). \tag{4}$$

The following explicit formulas are direct consequences of (1)–(4):

$$\widetilde{(F \circ G)}(x) = Z_F(\tilde{G}(x), \tilde{G}(x^2), \ldots), \qquad \widetilde{F'}(x) = \frac{\partial Z_F}{\partial x_1}(x, x^2, x^3, \ldots),$$

$$\overline{(F \circ G)}(x) = Z_F(\bar{G}(x), \bar{G}(x^2), \ldots), \qquad \overline{F'}(x) = \frac{\partial \Gamma_F}{\partial x_1}(x, x^2, x^3, \ldots).$$

The series F, $\tilde{F}$, $\bar{F}$, Z_F, and Γ_F have been computed for many elementary species. Table 1 gives a short table.

Given any species F and any integer $n \in \mathbb{N}$ we can extract a subspecies $F_n \subseteq F$ by collecting all those F-structures having an underlying cardinality n. If $F = F_n$ we say that F is *concentrated*,

n	A	$n!Z_A$	$n!\Gamma_A$
1	X	x_1	x_1
2	E_2	$x_1^2 + x_2$	$x_1^2 - x_2$
3	E_3	$x_1^3 + 3x_1x_2 + 2x_3$	$x_1 - 3x_1x_2 + 2x_3$
3	C_3	$2x_1^3 + 4x_3$	$2x_1^3 - 2x_3$
4	E_4	$x_1^4 + 6x_1^2x_2 + 8x_1x_3 + 3x_2^2 + 6x_4$	$x_1^4 - 6x_1^2x_2 + 8x_1x_3 + 3x_2^2 - 6x_4$
4	$E_4^{\pm}$	$2x_1^4 + 16x_1x_3 + 6x_2^2$	$2x_1^4 - 6x_2^2 - 8x_1x_3 + 12x_4$
4	$E_2 \circ E_2$	$3x_1^4 + 6x_1^2x_2 + 9x_2^2 + 6x_4$	$3x_1^4 - 6x_1^2x_2 - 3x_2^2 + 6x_4$
4	P_4^{bic}	$6x_1^4 + 18x_2^2$	$6x_1^4 - 18x_2^2 + 12x_4$
4	C_4	$6x_1^4 + 6x_2^2 + 12x_4$	$6x_1^4 - 6x_2^2$
4	$E_2 \circ X^2$	$12x_1^4 + 12x_2^2$	$12x_1^4 - 12x_2^2$

TABLE 2. Atomic species on less than 4 points and their index and asymmetry index series.

or *lives*, on the cardinality n. In the general situation, we obviously have the following canonical decomposition:

$$F = F_0 + F_1 + F_2 + \cdots + F_n + \dots . \tag{5}$$

The above canonical decomposition can be further refined by applying sums and products to fundamental 'building blocks' called *atomic species*. We recall that the atomic species constitute a countable set (working up to natural isomorphism)

$$\mathcal{A} = \{X, E_2, E_3, C_3, E_4, E_4^{\pm}, E_2 \circ E_2, P_4^{\mathrm{bic}}, C_4, E_2 \circ X^2, \dots\}$$

and are defined as being the irreducible species with respect to both sums '+' and products '$\cdot$'. Moreover, $\mathcal{A}$ is a 'graded set'

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \cdots \cup \mathcal{A}_n \cup \cdots$$

where $\mathcal{A}_n$ is the finite set consisting of all those atomic species that are concentrated on cardinality n. A complete description of $\mathcal{A}_n$ can be found in [5, 8, 15]. It is well known (by Yeh's Theorem [8, 15]) that each F_n in decomposition (5) can be written in a unique way as a polynomial (with coefficients in $\mathbb{N}$) in the atomic species that live on cardinalities $\leq n$. Stated differently, this means that we have the following half-ring isomorphism

$$\text{Species} \simeq \mathbb{N}[[X, E_2, E_3, C_3, E_4, E_4^{\pm}, E_2 \circ E_2, P_4^{\mathrm{bic}}, C_4, E_2 \circ X^2, \dots]] = \mathbb{N}[[\mathcal{A}]]$$

where Species denotes the half-ring of all the species (under the operations '+' and '$\cdot$' and where equality '=' means natural isomorphism).

EXAMPLE. For the species Gr of simple graphs, we have the unique atomic decomposition:

$$Gr(X) = 1 + X + 2E_2 + 2X{\cdot}E_2 + 2E_3 + 2X^2{\cdot}E_2 + 2X{\cdot}E_3 + 2E_2{\cdot}E_2 + 2E_2 \circ E_2 + E_2 \circ X^2 + 2E_4 + \cdots .$$

The universal ring $\mathbf{V}$ containing $\mathbb{N}[[\mathcal{A}]]$ is called the *ring of virtual species*. Every element in $\mathbf{V}$ can be represented as $F - G$ where F and G are two species. The ring $\mathbf{V}$ is isomorphic to $\mathbb{Z}[[\mathcal{A}]]$ and is closed for the combinatorial sums, products, substitutions and derivations.

Table 2 gives the index series and the asymmetry index series (polynomial, in fact) of each atomic species on $n \leq 4$ points.

2. General Explicit and Recursive Formulas

Consider the combinatorial equation $A = X{\cdot}E(A)$ which characterizes the species A of rooted trees. We get in a purely mechanical way the following classical result [3]

$$A(x) = xe^{A(x)}, \tilde{A}(x) = x \exp\left(\sum_{n\geq 1} \frac{\tilde{A}(x^n)}{n}\right), \bar{A}(x) = x \exp\left(\sum_{n\geq 1} (-1)^{n-1}\frac{\bar{A}(x^n)}{n}\right),$$

$$Z_A = x_1 \exp\left(\sum_{n\geq 1} \frac{(Z_A)_n}{n}\right), \Gamma_A = x_1 \exp\left(\sum_{n\geq 1} (-1)^{n-1}\frac{(\Gamma_A)_n}{n}\right).$$

The fundamental Otter-Robinson-Leroux [12, 14, 10] equation

$$\mathcal{A} + A^2 = A + E_2(A),$$

between the species A of rooted trees and the species $\mathcal{A}$ of ordinary trees, gives the following results

$$\mathcal{A}(x) = A(x) - \frac{1}{2}(A(x))^2, \quad \tilde{\mathcal{A}}(x) = \tilde{A}(x) - \frac{1}{2}(\tilde{A}(x))^2 + \frac{1}{2}\tilde{A}(x^2) \text{ (Otter [12])},$$

$$\bar{\mathcal{A}}(x) = \bar{A}(x) - \frac{1}{2}(\bar{A}(x))^2 - \frac{1}{2}\bar{A}(x^2) \text{ (Harary-Prins [3])},$$

$$Z_{\mathcal{A}} = Z_A - \frac{1}{2}(Z_A)^2 + \frac{1}{2}(Z_A)_2 \text{ (Robinson [14])}, \quad \Gamma_{\mathcal{A}} = \Gamma_A - \frac{1}{2}(\Gamma_A)^2 - \frac{1}{2}(\Gamma_A)_2.$$

3. R-enriched rooted trees and R-enriched trees

The species A_R of *R-enriched rooted trees* (Labelle 1981) is recursively characterized by the following combinatorial equation (i.e. natural isomorphism between species):

$$A_R = X{\cdot}R(A_R). \tag{6}$$

Depending on the choice of "enriching species" R, this definition includes : *ordinary rooted trees* ($R = E$), *cyclic rooted trees* ($R = 1 + C$), *binary rooted trees* ($R = 1 + E_2$), *plane rooted trees* ($R = L$), *oriented rooted trees* ($R = E^2$), and *permutation rooted trees* ($R = S$).

A variant to the notion of R-enriched rooted tree is that of *R-enriched tree*. It is a tree in which the set of "immediate neighbours" of each node is equipped with an R-structure. The species of R-enriched rooted trees is denoted by $\mathcal{A}_R$.

Lemma 1 (Labelle 1981). *The species $\mathcal{A}_R^\bullet = X\frac{d\mathcal{A}_R}{dX}$ of pointed R-enriched trees satisfies*

$$\mathcal{A}_R^\bullet = XR(A_{R'}),$$

where $R' = \frac{dR}{dX}$ and $A_{R'} = XR'(A_{R'})$ is the species of R'-enriched rooted trees.

Lemma 2. *The species $\mathcal{A}_R$ of R-enriched trees and the species $A_{R'}$ of R'-enriched rooted trees are related by the combinatorial equation*

$$\mathcal{A}_R + A_{R'}^2 = XR(A_{R'}) + E_2(A_{R'}). \tag{7}$$

THEOREM 1. *From equations (6) and (7) we obtain the following ten formulas:*

$$A_R(x) = xR(A_R(x)), \tag{8}$$

$$\widetilde{A_R(x)} = xZ_R(\widetilde{A_R}(x), \widetilde{A_R}(x^2), \widetilde{A_R}(x^3), \ldots), \quad Z_{A_R} = x_1 Z_R(Z_{A_R}), \tag{9}$$

$$\overline{A_R(x)} = x\Gamma_R(\overline{A_R}(x), \overline{A_R}(x^2), \overline{A_R}(x^3), \ldots), \quad \Gamma_{A_R} = x_1 \Gamma_R(\Gamma_{A_R}), \tag{10}$$

$$\mathcal{A}_R(x) = xR(A_{R'}(x)) - \frac{1}{2}(A_{R'}(x))^2, \tag{11}$$

$$\widetilde{\mathcal{A}_R}(x) = xZ_R(\widetilde{A_{R'}}(x), \widetilde{A_{R'}}(x^2), \widetilde{A_{R'}}(x^3), \ldots) - \frac{1}{2}(\widetilde{A_{R'}}(x))^2 + \frac{1}{2}\widetilde{A_{R'}}(x^2), \tag{12}$$

$$\overline{\mathcal{A}_R}(x) = x\Gamma_R(\overline{A_{R'}}(x), \overline{A_{R'}}(x^2), \overline{A_{R'}}(x^3), \ldots) - \frac{1}{2}(\overline{A_{R'}}(x))^2 - \frac{1}{2}\overline{A_{R'}}(x^2), \tag{13}$$

$$Z_{\mathcal{A}_R} = x_1 Z_R(Z_{A_{R'}}) - \frac{1}{2}(Z_{A_{R'}})^2 + \frac{1}{2}(Z_{A_{R'}})_2, \Gamma_{\mathcal{A}_R} = x_1 \Gamma_R(\Gamma_{A_{R'}}) - \frac{1}{2}(\Gamma_{A_{R'}})^2 - \frac{1}{2}(\Gamma_{A_{R'}})_2. \tag{14}$$

THEOREM 2. *Let A_R be the species of R-enriched rooted trees. Then, for every partition $\sigma = (\sigma_1, \sigma_2, \ldots)$ and every species F,*

$$\operatorname{coeff}_\sigma Z_{A_R} = \operatorname{coeff}_\sigma x_1 \prod_{k\geq 1} \left(1 - \frac{x_1 \partial Z_R/\partial x_1}{Z_R}\right)_k (Z_R)_k^{\sigma_k}, \tag{15}$$

$$\operatorname{coeff}_\sigma Z_{F(A_R)} = \operatorname{coeff}_\sigma Z_F \cdot \prod_{k\geq 1} \left(1 - \frac{x_1 \partial Z_R/\partial x_1}{Z_R}\right)_k (Z_R)_k^{\sigma_k}, \tag{16}$$

$$\operatorname{coeff}_\sigma \Gamma_{A_R} = \operatorname{coeff}_\sigma x_1 \prod_{k\geq 1} \left(1 - \frac{x_1 \partial \Gamma_R/\partial x_1}{\Gamma_R}\right)_k (\Gamma_R)_k^{\sigma_k}, \tag{17}$$

$$\operatorname{coeff}_\sigma \Gamma_{F(A_R)} = \operatorname{coeff}_\sigma \Gamma_F \cdot \prod_{k\geq 1} \left(1 - \frac{x_1 \partial \Gamma_R/\partial x_1}{\Gamma_R}\right)_k (\Gamma_R)_k^{\sigma_k}. \tag{18}$$

THEOREM 3. *Let $\mathcal{A}_R$ be the species of R-enriched trees. Then, for every partition $\sigma = (\sigma_1, \ldots)$,*

$$\operatorname{coeff}_\sigma Z_{\mathcal{A}_R} = \begin{cases} \omega_{\sigma_1 - 1, \sigma_2, \sigma_3, \ldots} & \text{if } \sigma_1 \neq 0, \\ 2^{(\sum \sigma_{2k}) - 1} b_{\sigma_2, \sigma_4, \ldots} & \text{if } 0 = \sigma_1 = \sigma_3 = \cdots, \\ 0 & \text{otherwise}, \end{cases} \tag{19}$$

$$\operatorname{coeff}_\sigma \Gamma_{\mathcal{A}_R} = \begin{cases} \omega^\star_{\sigma_1 - 1, \sigma_2, \sigma_3, \ldots} & \text{if } \sigma_1 \neq 0, \\ 2^{(\sum \sigma_{2k}) - 1} b^\star_{\sigma_2, \sigma_4, \ldots} & \text{if } 0 = \sigma_1 = \sigma_3 = \cdots, \\ 0 & \text{otherwise}, \end{cases} \tag{20}$$

where

$$\omega_\sigma = \text{coeff}_\sigma\, Z_R \cdot \prod_{k\geq 1}\left(1-\frac{x_1\partial^2 Z_R/\partial x_1^2}{\partial Z_R/\partial x_1}\right)_k (\partial Z_R/\partial x_1)_k^{\sigma_k},$$
$$b_\sigma = \text{coeff}_\sigma\, x_1 \prod_{k\geq 1}\left(1-\frac{x_1\partial^2 Z_R/\partial x_1^2}{\partial Z_R/\partial x_1}\right)_k (\partial Z_R/\partial x_1)_k^{\sigma_k},$$
$$\omega_\sigma^\star = \text{coeff}_\sigma\, \Gamma_R \cdot \prod_{k\geq 1}\left(1-\frac{x_1\partial^2 \Gamma_R/\partial x_1^2}{\partial \Gamma_R/\partial x_1}\right)_k (\partial \Gamma_R/\partial x_1)_k^{\sigma_k}$$
$$b_\sigma^\star = \text{coeff}_\sigma\, x_1 \prod_{k\geq 1}\left(1-\frac{x_1\partial^2 \Gamma_R/\partial x_1^2}{\partial \Gamma_R/\partial x_1}\right)_k (\partial \Gamma_R/\partial x_1)_k^{\sigma_k}.$$

4. Examples

Every formula developed above can be implemented on symbolic computation systems such as MAPLE, MATHEMATICA, MACSYMA, or DARWIN (Bergeron 1988). Examples of computation are given in [7]. In the sequence, this paragraph contains concrete applications of some of our results for particular choices of the enriching species R.

$(\mathbf{R = 1 + C})$.

# of types of planes trees on n vertices =

$$\frac{1}{2(n-1)}\sum_{d|(n-1)}\phi\left(\frac{n-1}{d}\right)\binom{2d}{d} - \frac{1}{2}c_{n-1} + \frac{1}{2}\chi_{\text{even}}(n)c_{(n/2)-1},$$

# of types of asymmetric planes trees on n vertices =

$$\frac{1}{2(n-1)}\sum_{d|(n-1)}\mu\left(\frac{n-1}{d}\right)\binom{2d}{d} - \frac{1}{2}c_{n-1} - \frac{1}{2}\chi_{\text{even}}(n)c_{(n/2)-1},$$

where χ_{even} is the characteristic function of the set of even numbers, $\phi(n)$ and $\mu(n)$ respectively denote the classical Euler and Möbius functions of n, and $c_n = \frac{1}{n+1}\binom{2n}{n}$ are the usual Catalan numbers.

$(\mathbf{R = E})$. In the case $A_R = A_E = A$ (the species of rooted trees), (15) and (17) can be rewritten as

$$\text{coeff}_\sigma\, Z_A = \begin{cases} 0 & \text{if } \sigma_1 = 0, \\ \sigma_1^{\sigma_1 - 1}\prod_{k\geq 2}(\phi_k^{\sigma_k} - k\sigma_k\phi_k^{\sigma_k - 1}) & \text{otherwise,} \end{cases}$$
$$\text{coeff}_\sigma\, \Gamma_A = \begin{cases} 0 & \text{if } \sigma_1 = 0, \\ \sigma_1^{\sigma_1 - 1}\prod_{k\geq 2}(\theta_k^{\sigma_k} - k\sigma_k\theta_k^{\sigma_k - 1}) & \text{otherwise,} \end{cases}$$

where $\phi_k = \sum_{d|k} d\sigma_d$ and $\theta_k = \sum_{d|k}(-1)^{(k/d)-1}d\sigma_d$.

In the case $\mathcal{A}_R = \mathcal{A}_E = \mathcal{A}$ (the species of trees), (19) and (20) can be rewritten as

$$\mathrm{coeff}_\sigma\, Z_{\mathcal{A}_R} = \begin{cases} a_\sigma/\sigma_1 & \text{if } \sigma_1 \neq 0, \\ 2^{(\sum \sigma_{2k})-1} a_{\sigma_2,\sigma_4,\ldots} & \text{if } 0 = \sigma_1 = \sigma_3 = \cdots, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{coeff}_\sigma\, \Gamma_{\mathcal{A}_R} = \begin{cases} a^\star_\sigma/\sigma_1 & \text{if } \sigma_1 \neq 0, \\ 2^{(\sum \sigma_{2k})-1} a^\star_{\sigma_2,\sigma_4,\ldots} & \text{if } 0 = \sigma_1 = \sigma_3 = \cdots, \\ 0 & \text{otherwise,} \end{cases}$$

where $a_\sigma = \mathrm{coeff}_\sigma\, Z_A$ and $a^\star_\sigma = \mathrm{coeff}_\sigma\, \Gamma_A$.

For a direct application of (16) and (18), consider the species of *endofunctions* $End = S(A)$, where S is the species of permutations. Taking $F = S$ and $R = E$ in (16) and (18), a few computations give

$$\mathrm{coeff}_\sigma\, Z_{End} = \sigma_1^{\sigma_1} \prod_{k\geq 2} (\phi_k^{\sigma_k} - k\sigma_k \phi_k^{\sigma_k - 1})$$

$$\mathrm{coeff}_\sigma\, \Gamma_{End} = \sigma_1^{\sigma_1} (\theta_2^{\sigma_2} - 4\sigma_2 \theta_2^{\sigma_2 - 1} + 4\sigma_2(\sigma_2 - 1)\theta_2^{\sigma_2 - 2}) \prod_{k\geq 3} (\theta_k^{\sigma_k} - k\sigma_k \theta_k^{\sigma_k - 1}),$$

where $\phi_k = \sum_{d|k} d\sigma_d$ and $\theta_k = \sum_{d|k} (-1)^{(k/d)-1} d\sigma_d$.

($\mathbf{R} = \mathbf{S}$). The A_S-structures are called *permutation rooted trees.* In this case, formula (10) takes the very compact form

$$\overline{A_S}(x) = \sum_{n\geq 0} \bar{a}_n x^n = x \frac{1 - \overline{A_S}(x^2)}{1 - \overline{A_S}(x)},$$

where $\bar{a}_0 = 0$, $\bar{a}_1 = 1$, and $\bar{a}_{n+1} = (\bar{a}_1\bar{a}_n + \bar{a}_2\bar{a}_{n-1} + \cdots + \bar{a}_n\bar{a}_1) - \chi_{\mathrm{even}}(n)\bar{a}_{n/2}$.

($\mathbf{R} = \mathbf{E} - \mathbf{E}_2$). A *topological tree* (also called *homeomorphically irreducible tree*) is a tree that has no node of degree 2. The species $\mathcal{A}_{top}$ of topological trees can be expressed in terms of the species $\mathcal{A}$ and A through the combinatorial equation

$$\mathcal{A}_{top} = \mathcal{A}\left(\frac{X}{1+X}\right) + XA\left(\frac{X}{1+X}\right) - XE_2\left(A\left(\frac{X}{1+X}\right)\right).$$

This equation gives the formulas

$$\begin{aligned} \widetilde{\mathcal{A}_{top}}(x) &= Z_{\mathcal{A}}\left(\frac{x}{1+x}, \frac{x^2}{1+x^2}, \ldots\right) + xZ_A\left(\frac{x}{1+x}, \frac{x^2}{1+x^2}, \ldots\right) \\ &\quad - \frac{x}{2} Z_A^2\left(\frac{x}{1+x}, \frac{x^2}{1+x^2}, \ldots\right) - \frac{x}{2} Z_A\left(\frac{x^2}{1+x^2}, \frac{x^4}{1+x^4}, \ldots\right), \\ \overline{\mathcal{A}_{top}}(x) &= \Gamma_{\mathcal{A}}\left(\frac{x}{1+x}, \frac{x^2}{1+x^2}, \ldots\right) + x\Gamma_A\left(\frac{x}{1+x}, \frac{x^2}{1+x^2}, \ldots\right) \\ &\quad - \frac{x}{2} \Gamma_A^2\left(\frac{x}{1+x}, \frac{x^2}{1+x^2}, \ldots\right) + \frac{x}{2} \Gamma_A\left(\frac{x^2}{1+x^2}, \frac{x^4}{1+x^4}, \ldots\right). \end{aligned}$$

5. Related topics

THEOREM 4. *The number of types of rooted plane trees with degree distribution* $\vec{\imath} = (i_1, i_2, i_3, \dots)$ *is*

$$\frac{1}{n-1} \sum_{p \in supp\ \vec{\imath}} \sum_{d | p, \vec{\imath} - \vec{1}_p} \phi(d) \binom{(n-1)/d}{i_1/d, i_2/d, \dots, (i_p - 1)/d, \dots},$$

where $supp\ \vec{\imath} = \{p \mid i_p \neq 0\}$, $d \mid \vec{\imath}$ *iff* $\forall p \ :\ d \mid i_p$, *and* $\vec{\imath} - \vec{1}_p = (i_1, i_2, \dots, i_p - 1, \dots)$.

THEOREM 5. *The number of types of bicoloured plane trees with degree distributions* $\vec{\imath} = (i_1, i_2, i_3, \dots)$ *and* $\vec{\jmath} = (j_1, j_2, j_3, \dots)$ *is*

$$\frac{1}{n} \sum_{p \in supp\ \vec{\jmath}} \sum_{d | p, \vec{\imath}, \vec{\jmath} - \vec{1}_p} \phi(d) \binom{n/d}{i_1/d, i_2/d, \dots} \binom{(m-1)/d}{j_1/d, j_2/d, \dots, (j_p - 1)/d, \dots}$$
$$+ \frac{1}{m} \sum_{p \in supp\ \vec{\imath}} \sum_{d | p, \vec{\imath} - \vec{1}_p, \vec{\jmath}} \phi(d) \binom{(n-1)/d}{i_1/d, i_2/d, \dots, (i_p - 1)/d, \dots} \binom{m/d}{j_1/d, j_2/d, \dots}$$
$$- \frac{n+m-1}{nm} \binom{n}{i_1, i_2, \dots} \binom{m}{j_1, j_2, \dots},$$

where $supp\ \vec{\imath} = \{p \mid i_p \neq 0\}$, $d \mid \vec{\imath}$ *iff* $\forall p \ :\ d \mid i_p$, *and* $\vec{\imath} - \vec{1}_p = (i_1, i_2, \dots, i_p - 1, \dots)$.

Let $B = B(X, Y)$ be the species of rooted trees with internal point of sort X and leaves of sort Y. This species is characterized by the functional equation

$$B = Y + X \cdot E^{\star}(B)$$

where $E^{\star} = E - 1$ stands for species characteristic of nonempty sets.

THEOREM 6. *The species* $A = A(X)$ *and* $B = B(X, Y)$ *are related by the following combinatorial equation*

$$B = Y - X + A(X \cdot E(Y - X))$$

where $E = E(X)$ *is the species of sets.*

Let $\mathcal{B} = \mathcal{B}(X, Y)$ be the species of trees with internal point of sort X and leaves of sort Y.

THEOREM 7. *The species* $\mathcal{A} = \mathcal{A}(X)$ *and* $\mathcal{B} = \mathcal{B}(X, Y)$ *are related by the following combinatorial equation*

$$\mathcal{B} = (Y - X) + E_2(Y - X) + \mathcal{A}(X \cdot E(Y - X))$$

where $E = E(X)$ *is the species of sets and* $E_2 = E_2(X)$ *is the species of sets of cardinality two.*

THEOREM 8. *Let* U *be an* n*-set,* σ *be a permutation of* U *whose cyclic type is* $(\sigma_1, \sigma_2, \dots, \sigma_n)$. *Then, for* $n \geq 2$, *the expected number of leaves in a random rooted tree (resp. in a random tree) on* U *of which* σ *is an automorphism is given respectively by*

$$\frac{1}{a_\sigma} \sum_{k=1}^{n} k \sigma_k \left(\sum_{d|k} d \sigma_d - k \right) \cdot a_{(\sigma_1, \dots, \sigma_k - 1, \dots, \sigma_n)}$$
$$\frac{1}{\alpha_\sigma} \sum_{k=1}^{n} k \sigma_k \left(\sum_{d|k} d \sigma_d - k \right) \cdot \alpha_{(\sigma_1, \dots, \sigma_k - 1, \dots, \sigma_n)}$$

where a_σ (resp. α_σ) is the number of rooted trees (resp. trees) of which σ is an automorphism.

Example. The expected number of leaves in a random rooted tree with σ as automorphism is

(1) $n(n-1)^{n-1}/n^{n-1} \sim n/e$ if $\sigma = \mathrm{Id}_n$ (well known),

(2) $\frac{(n-3)^{n-2}}{(n-2)^{n-3}} + 2$ if σ is of type $(n-2, 1, 0, \ldots, 0)$,

(3) 97.89276140 if $n = 186$ and σ is of type $(6, 1, 12, 0, 0, 0, 4, 3, 2, 0, 0, 6)$ (example given after a few seconds, using Maple on a personal computer).

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