Analysis of families of polynomials

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[summary by Bruno Salvy]

Abstract

When the generating function of a family of polynomials $P_n(x)$ can be handled by singularity analysis, it is possible to obtain quantitative results about the localization of the roots of these polynomials. The computation goes through three main steps: *i)* computation of the singularities when x is fixed as functions of x; *ii)* application of a uniform version of singularity analysis at these singularities; *iii)* matching the singular expansions to get an equation relating the roots x_n and n. One can then obtain an error bound on the asymptotic estimate of the zeros by Rouché's theorem.

The talk describes the use of this method on four examples displaying different cases corresponding to different types of singularities. In all cases, the generating function has an explicit form looking like

$$P(x,z) = \sum_{n\geq 0} P_n(x) z^n = \frac{1}{1 - xa(z)},$$

where a is an analytic function which is 0 at the origin.

1. Two simple poles

This first example, $a(z) = z + z^2$ shows the outline of the method but does not require singularity analysis.

(1) When x is fixed, the singularities are at the two roots of the denominator, namely $z_{\pm} = -1/2(1 \pm \sqrt{1 + 1/4x})$;
(2) Since the poles are simple (except when $x = -1/4$), one gets the exact value $P_n(x) = \frac{z_+^{-n-1}}{1-x(1+2z_+)} + \frac{z_-^{-n-1}}{1-x(1+2z_-)}$.
(3) For x to be a zero, it is necessary that the terms in the above sum cancel. A simple computation shows that this happens only when $z_+^{n+1} = z_-^{n+1}$. In other words, z_+/z_- has to be a $n+1$st root of unity. After resolution this yields the roots $x = -4\cos^2(k\pi/(n+1))$.

In this example, it would have been possible to be more direct by noticing that the polynomials P_n are related to Tchebychev polynomials of the second kind.

2. A fixed algebraic singularity and a moving pole

In this example $a(z) = 1 - \sqrt{1-z}$.

(1) Obviously $z = 1$ is a singularity, while the cancellation of the denominator yields a polar singularity at $z_1 = 1 - (1 - 1/x)^2$.

(2) Fixing some $\delta > 0$ for uniformity, one gets the local expansion at 1 as

$$P(x,z) = \frac{1}{1-x} - \frac{x}{(1-x)^2}\sqrt{1-z} + \frac{x^2}{(1-x)^3}(1-z) + O[(1-z)^{3/2})],$$

where the $O()$ is uniform in $|1-x| \geq \delta$.
At z_1, the residue is $2\frac{1-x}{x^2}$.
Singularity analysis then yields

$$P_n(x) = 2\left(\frac{x-1}{x^2}\right) z_1^{-n-1} + \frac{x}{(1-x)^2}\frac{n^{-3/2}}{2\sqrt{\pi}}[1+O(1/n)],$$

where the $O()$ is uniform in $|x| \geq \delta$, $|1-x| \geq \delta$.

(3) Cancellation of the leading terms yields

$$z_1^{-n-1} = \frac{n^{-3/2}}{4\sqrt{\pi}}\frac{x^3}{(1-x)^3}[1+O(1/n)],$$

and this can be solved asymptotically. One gets at the first order

$$x = \frac{1}{1-\sqrt{1-e^{i\theta}}}, \quad \theta = \frac{2k\pi}{n+1}, \quad |x| \geq \delta.$$

This shows a limit curve to which the roots tend regularly.

Rouché's theorem makes it possible to show that this first order estimate is accurate up to $O(1/n)$.

3. An infinity of poles

We now consider $a(z) = e^z - 1$. Properly normalized, the polynomials P_n have Stirling numbers of the second kind as coefficients.

(1) Cancelling the denominator gives the simple poles $z_k = \log(1+1/x) + 2ik\pi$.
(2) The residue is $1/(1+x)$.
(3) Cancellation of the dominant contributions implies $z_0 = \overline{z_1}$, and as a consequence, $1+1/x$ must be real negative. Solving the first order cancellation yields

$$x_k = -\frac{1}{1+\exp\left[-\pi\cot\left((k+1/2)\frac{\pi}{n+1}\right)\right]}.$$

Here also, an argument based on Rouché's theorem gives an error bound, and using a recurrence on the P_n one shows that the roots are not only asymptotically real, but actually lie in the interval $]-1,0]$. This last property implies that the sequence of coefficients of P_n is unimodal.

4. A moving pole and a fixed logarithmic singularity

The following generating function originates in a question by P. Curtz [1]:

$$P(x,z) = \frac{\log(1+z)}{z[1-x\log(1+z)]}.$$

The coefficients can be shown to be related to Stirling numbers from which integral representations of the generating function can be deduced and this gives another means to analyze the zeroes. Using the same method as previously, we get

(1) A fixed singularity at $z=-1$ and a pole at $z_1 = e^{1/x}-1$;

(2) Summing the contribution obtained by singularity analysis at -1 and by residue computation at z_1 one gets that
$$P_n(x) = \frac{e^{1/x}}{x^2} z_1^{-n-2} + \frac{(-1)^n}{x^2} \frac{1}{n \log^2 n} [1 + O(1/\log n)],$$
uniformly for $|x| \geq \delta$;

(3) Cancellation gives
$$x = \frac{1}{\log(1 + e^{i\theta})}, \quad \theta = \frac{2k\pi}{n+2}.$$

Once again the roots accumulate regularly around a limiting curve and an error bound can be obtained by Rouché's theorem.

Conclusion

This method seems to be of wide application. It is possible to extend it in order to get a full asymptotic expansion of the roots. It is also possible to study the case of families of polynomials whose generating function can be treated by a saddle-point method. Another extension leads to a quantitative version of Jentzsch's theorem about the roots of truncatures of Taylor series.

Bibliography

[1] Flajolet (Philippe), Gourdon (Xavier), and Salvy (Bruno). – Sur une famille de polynômes issus de l'analyse numérique. *Gazette des Mathématiciens*, vol. 55, January 1993, pp. 67–78.