A class of formal power series helps enumerate Young paths

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[summary by Dominique Gouyou-Beauchamps]

Abstract

We study different aspects of the enumeration of standard paths in the poset of compositions of integers. We show that many problems similar to those considered in the poset of partitions of an integer become simpler in this context. We give many explicit formulas for generating functions of standard paths in this poset and interesting subposets.

1. Standard Young Tableaux and Paths in the Young Lattice

A *partition* of a positive integer n is a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ such that $\sum_i \lambda_i = n$. We write $\lambda \vdash n$ to express this fact, and we say that k is the *height* $h(\lambda)$ of λ. The Ferrer diagram of a partition is the set of points $(i,j) \in \mathbb{Z}^2$ such that $1 \leq j < \lambda_i$.

A Young standard tableau T is an injective labelling of a Ferrer diagram by the elements of $\{1, 2, \ldots, n\}$, such that $T(i,j) < T(i+1,j)$, for $1 \leq i < k$, and $T(i,j) < T(i,j+1)$, for $1 \leq j < \lambda_i$. We further say that λ is the *shape* of the tableau T. For a given λ, the number f_λ of tableaux of shape λ is given by the *hook length formula*

$$f_\lambda = \frac{n!}{\prod_c h_c},$$

where $c = c(i,j)$ runs over the set of points in the diagram of λ, and

$$h_c = \lambda_i + \#\{j \mid \lambda_j \geq i\} - i - j + 1.$$

Other classical results in this context are $\sum_{\lambda \vdash n} f_\lambda^2 = n!$, and $\sum_{\lambda \vdash n} f_\lambda = \text{coeff of } \frac{x^n}{n!} \text{ in } e^{x+x^2/2}$.

We are interested in the enumeration of tableaux of height bounded by some integer h. In other words, we want to compute the numbers

$$t_h(n) = \sum_{h(\lambda) \leq h} f_\lambda,$$

and the series

$$y_h(x) = \sum_{n \geq 0} t_h(n) x^n.$$

Closed formulas for $t_h(n)$ are known for $n \leq 5$. Regev [6] has given asymptotic values for these numbers. The series $y_h(x)$ are differentiably finite (see Stanley [7]) (i.e. the $t_h(n)$'s are P-recursive). This means that the $t_h(n)$'s satisfy a recurrence of the form

$$\sum_{k=0}^{m} p_k(n) t_h(n-k) = 0,$$

for some polynomials $p_k(n)$ and some integer m.

Conjecture 1. [1]

(1) *the $t_h(n)$'s satisfy a recurrence of the form*

$$\sum_{k=0}^{\lfloor h/2\rfloor+1} p_k(n)t_h(n-k)=0, \tag{1}$$

for some polynomials $p_k(n)$ each of degree $\leq \lfloor h/2\rfloor$,

(2) *the coefficient of $t_h(n)$ in (1) is*

$$p_0(n)=\prod_{k=1}^{\lfloor h/2\rfloor}(n+k(h-k)),$$

(3) *for odd h, the coefficient of $t_h(n-1)$ in (1) is*

$$-p_1(n)=np_0(n)-(n-1)p_0(n-1),$$

(4) *for $h=7$ we have*

$$\begin{aligned}(n+6)(n+10)(n+12)t_7(n) =&(4n^3+78n^2+424n+495)t_7(n-1)\\ &+(n-1)(34n^2+280n+305)t_7(n-2)\\ &-(n-1)(n-2)(76n+290)t_7(n-3)\\ &-105(n-1)(n-2)(n-3)t_7(n-4).\end{aligned}$$

2. Compositions of n

Let us recall that a *composition* P is a sequence of positive integers $(p_i)_{i=1,\ldots,k}$. The p_i's are called *parts* of the composition and k, the number of parts, is said to be the *length* $\ell(P)$ of P and is denoted by $\ell(P)$. The *weight* $|P|$ of a composition P is the sum of its parts

$$|P|=\sum_{i=1}^{k}p_i=n.$$

We often say that P is a composition of n and write $P\models n$. The partition obtained by reordering the parts of a composition P is denoted $\lambda(P)$.

We say that a composition Q covers a composition P, if Q is obtained either by adding 1 to a part of P, or by adding a part of size 1 to P. The partial order obtained by transitive closure of this covering relation is denoted

$$P\prec R,$$

and the poset thus obtained is denoted Γ. For partitions, the analogous order corresponds to the inclusion of Ferrer diagrams. The poset of partitions is denoted Λ and the function $\lambda:\Gamma\longrightarrow\Lambda$, defined above, is a morphism of (graded) posets.

Our first objective will be the enumeration, with some parameters, of "standard" (up-going) paths starting with the composition (1) and finishing at $P\models n$. We also consider this enumeration problem for subposets obtained by restrictions on the compositions. More precisely, a *standard path* is a sequence of compositions

$$(1)=P_1\prec P_2\prec P_3\prec\cdots\prec P_n=P,$$

where $P_i\models i$. Such a path P is said to have *length* n, and we denote it $|\mathcal{P}|$. A standard path $\mathcal{P}=P_1\prec P_2\prec P_3\prec\cdots\prec P_n$ with endpoint P can be encoded by a permutation $\sigma(\mathcal{P})$ in the following way. We form a sequence of words

$$(1)=\omega_1,\omega_2,\ldots,\omega_n=\sigma(\mathcal{P})$$

where ω_i is obtained from ω_{i-1} by insertion of i in position $j=p_1^{(i)}+p_2^{(i)}+\cdots+p_k^{(i)}$ if P_i is obtained from P_{i-1} by adding 1 to the k-th part of P_{i-1}, in position $j=p_1^{(i)}+p_2^{(i)}+\cdots+p_k^{(i)}-1$ if P_i is obtained by

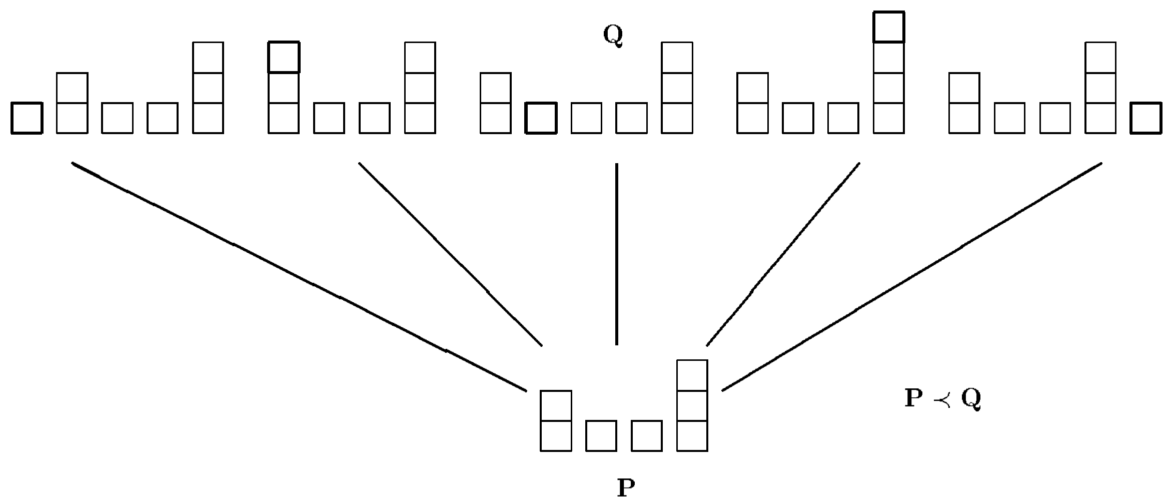


Figure 1

adding a part of size 1 to P_{i-1}, just after a part of size > 1, and in first position if the new part is added at the beginning of P_{i-1}.

For example, to the path

$$\mathcal{P} = (1) \prec (1,1) \prec (2,1) \prec (1,2,1) \prec (2,2,1) \prec (2,3,1) \prec (2,4,1) \prec (2,4,2) \prec (2,4,1,2)$$

there corresponds the sequence of words

$$1, 21, 231, 4231, 45231, 452361, 4523671, 45236718, 452369718$$

hence $\sigma(\mathcal{P}) = 452369718$.

Let us denote $P(\omega)$ the composition encoding the descents of a permutation ω

$$P(\omega) = (p_1, p_2, \ldots, p_k).$$

This means that the set $\{i \mid \omega_i > \omega_{i+1}\}$ coincides with the set $\{p_1, p_1 + p_2, p_1 + p_2 + p_3, \ldots\}$. Then

$$P(\sigma(\mathcal{P})) = \mathcal{P}, \qquad \text{and} \qquad \sigma(P(\omega)) = \omega,$$

if $\omega = \sigma(\mathcal{P})$ for some standard path $\mathcal{P}$. In order to unfold our study, we will also need the following alternative encoding of a standard path. First, we may formally define the *diagram* of a composition P to be the set of points $(i, j) \in \mathbb{Z}^2$ such that $1 \le j \le p_i$. It is convenient to replace the node (i, j) by the square with corners $(i-1, j-1)$, $(i-1, j)$, $(i, j-1)$ and (i, j). For a standard path ending at P, we label the squares of the diagram of P in the order of their apparition in the path. Thus the step

$$(2,3,1,5) \prec (2,4,1,5)$$

is encoded by the addition of the box labelled 12 in Figure 3.

The labelled diagram obtained in this manner is called the *tableau* of the path, and the underlying diagram (or composition) of a tableau is called its *shape*. This representation suggests that the number of parts of the endpoint P of a standard path $\mathcal{P}$ should be called the *width* of the path, the largest part the *height* of the path, and P the *shape* of the path.

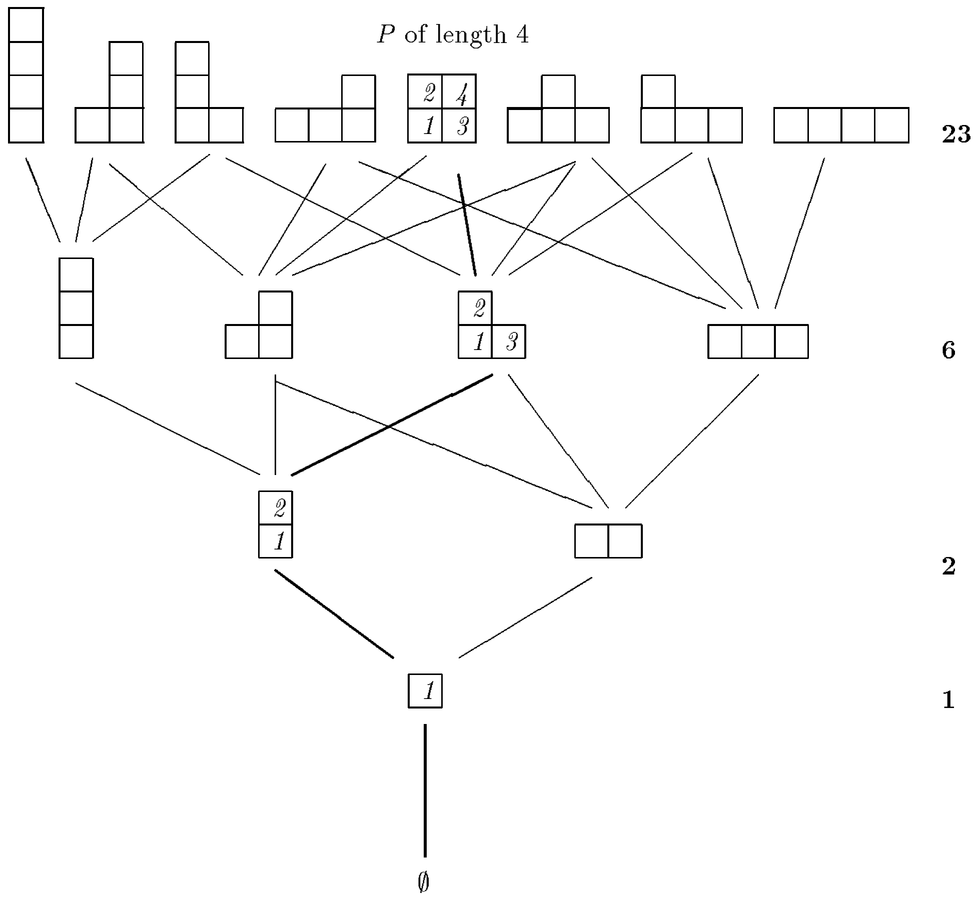


FIGURE 2

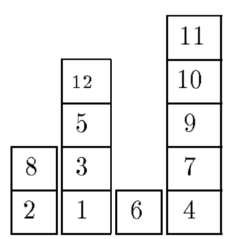


FIGURE 3.

3. Standard paths in the poset Γ

Let $\Gamma_{n,k,d}$ be the set of compositions of length n with k parts of size 1 and d parts of size > 1. Let $\gamma_{n,k,d}$ be the number of standard paths with endpoint in $\Gamma_{n,k,d}$. We would like to derive an explicit expression for the generating function

$$F(u,v,x) = \sum_{n\geq 1} \left(\sum_{k,j} \gamma_{n,k,d} u^k v^d \right) \frac{x^n}{n!}.$$

Examination of the different cases involved in the last step of a standard path gives the following recurrence

$$\gamma_{n+1,k,d} = d\gamma_{n,k,d} + (1+d)\gamma_{n,k-1,d} + (1+k)\gamma_{n,k+1,d-1},$$

with initial conditions $\gamma_{n,n,0} = 1$ and $\gamma_{n,0,1} = 1$, for $n \geq 2$. This recurrence translates into a partial differential equation for F

$$\frac{\partial}{\partial x} F(u,v,x) = (1+u)v\frac{\partial}{\partial v} F(u,v,x) + uF(u,v,x) + v\frac{\partial}{\partial u} F(u,v,x), \tag{2}$$

with initial conditions $F(u,0,x) = \exp(ux)$, and $(\frac{\partial}{\partial v} F(0,v,x))\big|_{v=0} = \exp(x) - 1 - x$. It is straightforward to verify (with the help of Maple) that the following function satisfies equation (2) with the prescribed initial conditions

$$F(u,v,x) = \frac{\alpha^2}{\exp(x)\left((1+u)\sin(\frac{\alpha}{2}x) - \alpha\,\cos(\frac{\alpha}{2}x)\right)^2}, \tag{3}$$

where

$$\alpha = \sqrt{2\,v - (1+u)^2}.$$

The first few terms of this series in x are

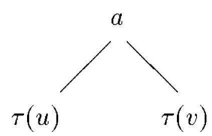
$$1 + ux + (v+u^2)\frac{x^2}{2!} + (v + 4\,vu + u^3)\frac{x^3}{3!} + (v + 4\,v^2 + 6\,vu + 11\,vu^2 + u^4)\frac{x^4}{4!}$$
$$+(v + 14\,v^2 + 34\,uv^2 + 8\,vu + 23\,u^2v + 26\,vu^3 + u^5)\frac{x^5}{5!} + \cdots.$$

It is not clear how one can come up with an expression such as (3) for the desired generating function. The following combinatorial argument describes one way of finding this expression.

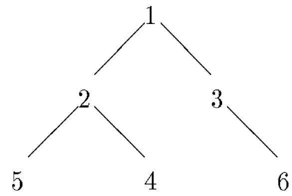
4. Increasing binary trees

First, we describe a classical bijection between permutations and *increasing binary trees*. For any word $w = w_1 w_2 \cdots w_n$ on $n \geq 1$ distinct letters (in an ordered alphabet), we recursively define the binary tree $\tau(w)$ to be

$\tau(w) =$



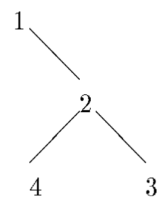
where $a = \min(w)$ is the minimum letter in w, u and v are the factors of w such that $w = uav$, $\tau(u)$ is the left branch of the tree, and $\tau(v)$ is the right branch. If one of these factors is the empty word, we omit the corresponding branch. Hence for the permutation $\omega = 521436$ the corresponding tree is

$\tau(524136) =$ 

It is clear that the labels in such a tree will be in increasing order on any path going from the root to a leaf. Since τ is a bijection, there are $n!$ increasing trees with labels $\{1, 2, \ldots, n\}$.

We can characterize the increasing trees $T = \tau(\sigma(\mathcal{P}))$ corresponding to the permutation encoding $\sigma(\mathcal{P})$ of standard paths $\mathcal{P}$ by the condition that for any node ν appearing in the left subtree of another node, when ν has two sons the label of its left son is inferior to that of the right one.

The smallest increasing tree that is excluded with this condition is



Using this characterization and the results of [5], it is easy to check that $F = F(x)$ satisfies the following system of differential equations

(4) $$F' = F(1+G), \qquad F(0) = 1, \qquad G' = 1 + 2\,G + G^2/2, \qquad G(0) = 0.$$

This is how we first obtained the generating function (3) (with $u = v = 1$). A finer study of the properties of these trees allows for the generalization of (4) accounting for the parameters u and v. We obtain

(5) $$F' = F(1+G), \qquad F(u,v,0) = 1, \qquad G' = v + (1+u)G + G^2/2, \quad G(u,v,0) = 0,$$

where $F'(x) = \frac{\partial}{\partial x} F(u,v,x)$.

The particular form of system (5) underlines that F is a *constructible differentially algebraic* series in the sense of [2]. Recall that a series $y = y(x)$, with coefficients in K, is said to be constructible differentially algebraic (CDF for short) if for some $k \geq 1$, there exist k series $y_1, \ldots, y_k$ with $y_1 = y$ and polynomials $P_1, \ldots, P_k$ (with coefficients in K) such that

$$\begin{aligned} y_1' &= P_1(y_1, \ldots, y_k) \\ &\vdots \\ y_k' &= P_k(y_1, \ldots, y_k) \end{aligned}$$

The class of CDF series contains polynomials, algebraic series, and the series expansion around 0 of the usual functions such as e^x, $\log(1+x)$, or the trigonometric functions and their inverse. It is closed for the usual operations on series: sum, product, composition, derivation, integration, inversion $(1/y(x))$, and inversion for composition. However it is not closed under Hadamard product (term-wise product). All CDF series are analytic around 0, hence $\sum_n n! x^n$ is not CDF which shows that this class does not contain the class of D-finite series (see [7, 8]). The series expansion around 0 of $1/\cos(x)$ is not D-finite, but is CDF. Thus the class CDF is not contained in the class of D-finite series.

5. Standard paths of bounded height

In the sequel of this paper, we denote $\Gamma_{(k)}$ the subposet of compositions of width $\leq k$, and $\Gamma^{(k)}$ the subposet of compositions of height $\leq k$.

The story is very similar for the poset $\Gamma^{(2)}$. Let, once again, $\Gamma^{(2)}_{k,d}$ be the set of compositions with k parts of size 1 and d parts of size 2. Clearly this implies that the length of the path is $n = k + 2\,d$. As before, let $\gamma^{(2)}_{k,d}$ be the number of standard paths with endpoint in $\Gamma^{(2)}_{k,d}$. The basic recurrence in this case is

$$\gamma^{(2)}_{k,d} = (1+d)\gamma^{(2)}_{k-1,d} + (1+k)\gamma^{(2)}_{k+1,d-1}. \tag{6}$$

We could proceed as in the derivation of (3) to deduce from (6) that

$$F^{(2)}(u,v) = \frac{\beta^2}{\left(u\,\sin(\frac{\beta}{2}) - \beta\,\cos(\frac{\beta}{2})\right)^2},$$

where $\beta = \sqrt{2\,v - u^2}$.

For a general k, the study of $\Gamma^{(k)}$ becomes quite intricate. We do not know at this time what are the generating functions for the enumeration of standard paths in those instances. In the case $k = 3$ the first terms of the corresponding generating function are

$$1 + x + 2\frac{x^2}{2!} + 6\frac{x^3}{3!} + 22\frac{x^4}{4!} + 98\frac{x^5}{5!} + 514\frac{x^6}{6!} + 3086\frac{x^7}{7!} + 20890\frac{x^8}{8!} + 157398\frac{x^9}{9!} + \cdots.$$

6. Standard paths of given width

The (ordinary) generating functions for the number of paths of *width* at most 2 is the rational function

$$F_{(2)}(x) = \frac{x^2 + x^3}{(1-x)(1-2\,x)}.$$

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