Rational Solutions of Linear Difference and Differential Equations

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[summary by Bruno Salvy]

Abstract

The talk presents an algorithm due to S. Abramov that computes rational solutions of an equation of the type

$$a_d(n)\,u_{n+d} + a_{d-1}(n)\,u_{n+d-1} + \cdots + a_1(n)\,u_{n+1} + a_0(n)\,u_n = b(n), \tag{1}$$

where the coefficients are polynomials in n. This algorithm computes the solutions without performing any factorization, but only gcd computations. Thus it is rather independent of the ground field and in particular, it performs well when the ground field is some algebraic extension of $\mathbb{Q}$. An analogous algorithm also due to S. Abramov solves the differential case.

1. Description of the algorithm

Once some $P(n)/Q(n)$ has been substituted for u_n into Equation (1), it is natural to attempt to deduce some information from the study of the poles of the rational functions involved. However, this is made difficult by the possibility of roots of Q differing by an integer. The idea of S. Abramov's algorithm is to first compute an upper bound h on the integers differences of roots of possible solutions, and then compute the linear relation satisfied by u_n, u_{n+h}, u_{n+2h}, ... :

$$f_q(n)\,u_{n+qh} + f_{q-1}(n)\,u_{n+(q-1)h} + \cdots + f_1(n)\,u_{n+h} + f_0(n)\,u_n = g(n), \tag{2}$$

In this relation the poles cannot interfere any longer and thus the polynomials f_j have to cancel the denominators of the rational functions u_{n+qh}. In other words,

$$Q(n) \mid \gcd(f_0(n), f_1(n-h), \ldots, f_q(n-qh)).$$

Changing the unknown function by eliminating this gcd, one gets a linear recurrence equation that must have a polynomial solution. Such a solution can be found by undeterminate coefficients.

The main difficulty is to compute an upper bound for h. Substituting again $P(n)/Q(n)$ for u_n in (1), one gets that the maximal integer difference between two roots of $Q(n)$ has to be less that the maximal integer difference between two roots of a_0 and a_d, minus d.

Once this bound for h has been found, there only remains to construct Equation (2). This is done by rewriting all the u_{n+ih} for $0 \le i \le d$ in terms of u_n, u_{n+1}, ... ,u_{n+d-1}, and then performing a Gaussian elimination. Note that in this stage it is only necessary to compute with the homogeneous part of Equation (1).

2. Extension of the algorithm

Once a solution $U_1(n)$ has been found, it is possible to reduce the order of the equation by a change of variable. If a new solution $U_2(n)$ is then found, it corresponds to the solution $U_1 \sum U_2$ of the initial equation. Again the process can be applied and one gets a chain of iterated sums. Of course the process is bound to stop because only d independent solutions can be found.

It is not obvious whether the algorithm outlined above will find *all* the solutions of (1) that are iterated sums, regardless of the order in which the solutions are found and the equation reduced. This was proved to be true in [2].

Another type of extension is by looking for sequences V_n such that $\Delta^k V_n$ is a rational solution of (1)[1]. This is made easy by the fact that for such a k to exist, Equation (1) must have a polynomial solution of degree k. The proof of this is as follows: suppose $\Delta^k V_n$ is a rational solution of (1), while $\Delta^{k-1} V_n$ is not rational. Starting from (1) we can compute a linear difference equation satisfied by $\Delta^{k-1} V_n$. Let this equation be

$$h_p(n)\,\Delta^p u_n + h_{p-1}(n)\,\Delta^{p-1} u_n + \cdots + h_1(n)\,\Delta u_n + h_0(n)\,u_n = b(n).$$

Then if h_0 is not zero, one can rewrite $u_n = \Delta^{k-1} V_n$ as a rational function of $(\Delta^k V_n, \Delta^{k+1} V_n, \ldots)$, and since all these functions are rational, this would imply that $\Delta^{k-1} V_n$ is rational. Thus $h_0 = 0$, but then it means that $\Delta^{k-1} V_n = 1$ is a solution, and this implies that V_n is a polynomial of degree k. This reasoning also provides an algorithm.

Bibliography

[1] Abramov (S. A.). – Rational solutions of linear differential and difference equations with polynomial coefficients. *USSR Computational Mathematics and Mathematical Physics*, vol. 29, n° 11, 1989, pp. 1611–1620. – Translation of the *Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki.*

[2] Abramov (S. A.). – A problem in computer algebra connected with solutions of linear differential and difference equations. *Kibernetika*, vol. 2, 1991, pp. 30–37. – (Russian).

[1]Here Δ denotes the difference operator: $\Delta u_n = u_{n+1} - u_n$.