12

Functions in Symbolic Computation: Time to Move On

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[summary by Kevin J. Compton]

The existing approach to symbolic computation is based on elementary functions (built from 1, π, and variables like x using $+$, $-$, $\times$, $\div$, and functions exp, log, sin). Standard notation gives a way of representing such functions. Symbolic computation with elementary functions is highly developed.

- To test *zero equivalence* one uses the Risch-Norman method plus the assumption of Schanuel's conjecture to distinguish constants.
- *Differentiation* of elementary functions is easy; integration is hard, but much has been achieved.
- *Differential equations* are harder still, but there are some results.
- Determining *limits* is undecidable if unrestricted use of sine is allowed. (See Shackell [5].) However, if sine is excluded there is a theoretical basis for limits in some cases.
- For *zero isolation* (cylindrical decomposition) there is now an algorithm due to Richardson [1] but as yet no implementation.

There are a number of gaps to be filled but symbolic computation must still be reckoned a success. Nonetheless, Shackell believes that it is time for the main thrust of research in this area to start along different lines. There are several reasons for this.

First, the answer to a problem may not be given by an elementary function. In some cases we do not know whether the answer is elementary or not. This is a hard question. If it isn't elementary (or at least Liouvillian, see below) it cannot be expressed in standard systems for representing functions. Usually, a scientist or engineer is just concerned with getting an answer, not in beautiful mathematics: a proof that an elementary solution does not exist is not of interest.

Second, the system should provide qualitative answers. Stoutemyer has argued that the user needs help to interpret a complicated formula. The system should give information about zeros, singularities, limits, symmetries, and perhaps sketch a graph.

This leads to the question of whether we really need a formula at all. If not, do functions have to be elementary? The difficulty is that we need algorithms so we must have some type of representation.

Main Thesis. *Shackell suggests taking an algebraic differential equation approach. In this approach functions are built up using algebraic differential equations.*

For example, one might specify

$$f(x) = x^{-2}a_1'(x) + (x+1)a_2(x) + x^3,$$

where a_2 satisfies $y''y+(x+a_1(x))y'^2+x^2+3=0$, $y(0)=1$, $y'(0)=2$, and a_1 satisfies $y'''=x^3y'^2+y''$, $y(0)=0$, $y'(0)=2$, $y''(0)=3$. Thus, in this approach there is a tower of differential fields

$$\mathbb{R}(x)=\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n$$

such that $\mathcal{F}_{i+1}$ is a differential extension of $\mathcal{F}_i$ by a solution of an algebraic differential equation over $\mathcal{F}_i$. (In fact, any function in $\mathcal{F}_n$ satisfies an algebraic differential equation over $\mathbb{R}$, but there are advantages in using the tower).

Definition 1 *The functions obtained by taking all differential equations of the form $y'=r_i(y)$ with r_i a rational function over $\mathcal{F}_i$ are called* Pfaffian functions.

Definition 2 *The functions obtained by taking all differential equations either of degree 0 (algebraic extension) or of the form $y'=yf_i$ with $f_i \in \mathcal{F}_i$ (exponential extension) are called* Liouvillian functions.

Are these two classes of functions reasonable for symbolic computation?

- For *zero equivalence* there are algorithms modulo an oracle for constants. (See Shackell [6] for the Pfaffian case and general case.)
- *Integrals* and solutions of *differential equations* exist within the system automatically.
- *Zero isolation* can be done in the Pfaffian case and slightly beyond. (See Richardson [1].)
- Perhaps symmetries can be computed. The problem is reducible to zero isolation, so can be done in the Pfaffian case. This may work in a more general case.
- Limits in the Liouvillian case are more or less done modulo constants [3, 4]. Perhaps this is extendable to the Pfaffian case. There is some information in the more general case, but it is undecidable in full generality.
- Should symbolic integration be ignored? The integral of e^x should not be defined as the solution of the differential equation $y'=e^x$. The functions provided by the user (corresponding to the tower of differential fields) may have physical significance. It would be nice to give the solution as simply as possible in terms of these. Mike Singer [7] has done some relevant work for linear differential equations, but the problem is probably very hard in general.

Everything so far has been modulo a method for deciding the signs of constants. The Schanuel conjecture gives a method for elementary functions. What might we do in the more general case? Shackell suggests taking a numeric/symbolic approach in which we handle constants numerically but otherwise work symbolically. Algorithms would need modification to take account of approximation of coefficients. Answers would only be correct modulo a range of error and might need special interpretation. Partial answers may have to be accepted, but there may be no alternative. In practice one needs a fast method of determining the sign of a constant. Perhaps even where the purely symbolic approach is possible, the numeric/symbolic approach would be faster.

One might expect to get answers of the following type. For symmetry one might have that $f(a+x)=f(a-x)\pm\varepsilon g(x)$. For limits, expressions like $f_0(x)\sim(1\pm\varepsilon)e^x$ and $f_1(x)\sim e^{(1\pm\epsilon)x}$ should be

acceptable, but not $f_2(x) \sim 2x \pm \varepsilon^x$. (The asymptotic expression for f_2 might even be acceptable in some ranges.)
A purely symbolic approach for handling constants looks hard in general. The Schanuel conjecture says roughly that the relations between constants are the ones we already know. We know little about Pfaffian constants. New work by Richardson [2] gives some glimmers of hope but may not work out.
Shackell suggests that the future work is needed in the following areas.

- Solve the constants problem using a numeric/symbolic approach if necessary.
- Extend word on Liouvillian case to Pfaffian case.
- Check out the method for finding symmetries.
- Gain practical experience through pilot implementations.

What level of generality should we have in practical development? Shackell compares three alternatives: Liouvillian functions, Pfaffian functions, and arbitrary towers of algebraic differential equations. The first is not very adventurous but still might be worthwhile. The problem with the last is the scarcity of algorithms; there are only a zero equivalence modulo constants, a little on limits, and a little on symmetry. Pfaffian functions seem to offer the most hope. There are quite a number of algorithms for this class, there is some chance of generalizing the limit algorithm, and there is a better chance of finding a method to deal with constants than in the case of towers of algebraic differential equations. The biggest problem is that many functions that arise will not be Pfaffian.
In conclusion, the range of functions that can be handled by algebra systems is very important and there is a good chance for development here.

References

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