1

Enumeration of Semi–Standard Young Tableaux

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[summary by Philippe Flajolet]

Standard tableaux. A *Young tableau* (also known as a *standard tableau*) is an arrangement of distinct integers in an array of left justified rows, such that the entries of each row are in increasing order from left to right and the entries of each column are in increasing order from bottom to top[2]. For instance, here is an array of size 15:

11					
4	8	12	14		
3	6	7	13		
1	2	5	9	10	15.

In general a tableau of n cells has a shape described by a partition of the integer n, the shape of the example being $(6, 4, 4, 1)$.
Such tableaux originate in the study of linear representations of the symmetric group. The main result is the following: *Each permutation of* $[1..n]$ *is uniquely representable by a pair of tableaux of size* n *having the same shape.* The effective mapping—based on the "bumping rule"—constitutes the celebrated Robinson–Schensted correspondence. A very readable introduction to the subject is to be found in Knuth's book [4, Sec. 5.1.4].
Many properties result from the Robinson–Schensted correspondence. First, the total number of tableaux of size n is equal to the number of involutions (these are permutations τ such that $\tau^2 = Id$) of $[1..n]$, that is

$$T_n = n! \cdot [z^n] \exp(z + \frac{z^2}{2}).$$

Next, the length of the longest increasing subsequence of a permutation τ coincides with the common base size of the pair of tableaux associated with τ.
The enumeration of tableaux of bounded width or height thus appears of interest in connection with various order statistics, these structures being also rich combinatorial objects *per se*. Let $T_n^{[k]}$ be the number of tableaux of height $\le k$, and denote by $C_n = 1/(n+1)\binom{2n}{n}$ the Catalan number. Then, from the works of Regev and Gouyou–Beauchamps, we have

$$T_n^{[2]} = \binom{n}{\lfloor n/2 \rfloor}; \quad T_n^{[3]} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} C_i; \quad T_n^{[4]} = C_{\lfloor (n+1)/2 \rfloor} C_{\lceil (n+1)/2 \rceil}. \tag{1}$$

The formulæ become intricate as k increases. However, it is known that the quantities $T_n^{[k]}$ are holonomic for each fixed k. See Gessel's paper on this aspect of things [3].

[2]There is a concurrent tradition of representing such arrays downwards!

Semi–standard tableaux. This part deals with results obtained jointly by Gouyou–Beauchamps and Seul Hee Choi in a paper yet to appear [2]. They are relative to *semi–standard* tableaux. A semi–standard tableau is strictly increasing in rows but only weakly increasing in columns. Here is one:

$$\begin{array}{lllll} 4 \\ 2 & 3 & 5 \\ 1 & 2 & 4 \\ 1 & 2 & 3 & 4 & 5\,. \end{array} \tag{2}$$

B. Gordon had proved long ago (in the 1960's) that the number of such tableaux with at most r rows (height at most r) and entries between 1 and n admits a product form

$$a_{n,r} = \prod_{1\le i\le j\le n} \frac{r+i+j-1}{i+j-1}. \tag{3}$$

This looks much simpler than what (1) reveals of the corresponding problem for standard tableaux. Choi and Gouyou–Beauchamps obtain a new formula that refines on (3) by keeping track of the parity of columns.

Theorem 1 *The number of semi–standard Young tableaux with entries in* $[1..n]$ *having height at most* $2k$ *and having* p *columns of odd height is*

$$c_{n,2k,p} = \frac{\binom{n}{p}\binom{2k+p-1}{p}}{\binom{n+2k+p}{p}} \prod_{1\le i\le j\le n} \frac{2k+i+j}{i+j}.$$

The numbers $c_{n,2k,0}$ had been earlier determined by Desainte–Catherine and Viennot, and the work presented here constitutes a refinement of their method. The ingredients of the proof are as follows.
1. Semi–standard tableaux are bijectively equivalent to certain non negative matrices and to certain generalized involutions (Knuth 1970, Burge 1974).
2. Semi–standard tableaux are bijectively equivalent to special pilings of Dyck paths. (A Dyck path is a non negative gambler ruin sequence.)
3. The counting of the relevant Dyck path configurations is expressible in terms of determinants involving ballot numbers and Hankel determinants of Catalan numbers. For instance,

$$\begin{vmatrix} C_0 & C_1 & \cdots & C_{n-1} & C_n \\ C_1 & C_2 & \cdots & C_n & C_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{n-1} & C_n & \cdots & C_{2n-2} & C_{2n-1} \\ C_n & C_{n+1} & \cdots & C_{2n-1} & C_{2n} \end{vmatrix} = 1.$$

(This part involves the Gessel–Viennot theory of path determinants.)
4. The involved determinants can finally be calculated using Viennot's combinatorial theory of the qd algorithm.
The whole enterprise is a rather delicate (and intricate) piece of bijective combinatorics. The effort invested is also justified by a general attempt at understanding the combinatorics of Gordon's amazingly simple formula (1). Initially, it arose as a specialization of a q–analog¡, itself related to plane partitions—that is, generalized tableaux classified according to the sum of their elements. Andrews' book can be consulted on some of these aspects, see [1, Ch. 11]. Plane partitions are lurking in the background.

References

[1] G. E. Andrews. *The Theory of Partitions*, volume 2 of *Encyclopedia of Mathematics and its Applications*. Addison–Wesley, 1976.

[2] S. H. Choi and D. Gouyou-Beauchamps. Énumeration de tableaux de Young semi-standards. To appear in *Theoretical Computer Science*, 1992.

[3] I. M. Gessel. Symmetric functions and P–recursiveness. *Journal of Combinatorial Theory,* Series A, 53:257–285, 1990.

[4] D. E. Knuth. *The Art of Computer Programming*, volume 3 : Sorting and Searching. Addison-Wesley, 1973.