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Maxima in Convex Regions

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[summary by Pierre Nicodème]

Suppose that C is a bounded convex planar region. Let $p_1, \ldots, p_n$ be n points drawn independently identically distributed from the uniform distribution over C and let M_n^C be the number of the points which are maximal. We present results showing how the asymptotic behaviour of $\mathbf{E}\left(M_n^C\right)$ depends on the geometry of C.

1 Introduction

We discuss $\mathbf{E}\left(M_n^C\right)$, the expected number of maximal points (that is such that there is no other point in the North-East quadrant from this point) in a set of n Independently Identically Distributed (I.I.D.) points drawn from the uniform distribution over some convex bounded planar region C.
The corresponding question for convex hulls has been well studied. Renyi and Sulanke [7, 8] proved that if n points are chosen I.I.D. from C then, if C is a convex polygon, the expected number of convex hull points is $\Theta(\log n)$ while if C is convex and has a doubly continuously differentiable boundary the answer is $\Theta\left(n^{1/3}\right)$. Dwyer [4] provides a survey of more recent results.
The expected number of maxima has not been examined nearly as closely. It has been known for many years [1] that if C is the unit square then $\mathbf{E}\left(M_n^C\right) = \Theta(\log n)$. Recently Dwyer [4] proved that $\mathbf{E}\left(M_n^C\right) = \Theta(\sqrt{n})$ when C is a circle and also proved a general upper bound $\mathbf{E}\left(M_n^C\right) = O(\sqrt{n})$ that is valid for any bounded convex planar region C (this result can also be found in Devroye [3]).
In this paper we study the asymptotics of $\mathbf{E}\left(M_n^C\right)$ in detail. Note that the condition that C be convex is important. If it is abandoned then it can be shown that, for all functions f, $f(n) \leq n/\log^2 n$, and f slowly varying at infinity, there is some C such that $\mathbf{E}\left(M_n^C\right) = \Theta(f(n))$ (see [6] for details). If C is constrained to be convex the situation is very different. We show in this paper that for convex C either $\mathbf{E}\left(M_n^C\right) = \Theta(\sqrt{n})$ or $\mathbf{E}\left(M_n^C\right) = O(\log n)$; nothing between these two functions is possible. We also give sufficient conditions for $\mathbf{E}\left(M_n^C\right) = \Theta(1)$ and $\mathbf{E}\left(M_n^C\right) = \Theta(\log n)$.
The rest of the paper will use the following notation: if $p = (p.x, p.y)$ and $q = (q.x, q.y)$ are planar points we say that p *dominates* q if $p.x \geq q.x$ and $p.y \geq q.y$. If $S = \{p_1, \ldots, p_n\}$ is a set of points we say that p is *maximal* in S if there is no $q \in S$, $q \neq p$, such that q dominates p. We set

$$\mathrm{MAX}(S) = \{p : p \text{ is maximal in } S\}.$$

See Figures 1 (a), (b) and (c). Suppose C is a bounded planar region. Let $S = \{p_1, \ldots, p_n\}$ be a set of n points drawn I.I.D. from the uniform distribution over C. Let $M_n^C = |\mathrm{MAX}(S)|$ be the number of maximal points in S. We will study $\mathbf{E}\left(M_n^C\right)$, the expected number of maximal points. We will also look at the *outer layer* of S, the set of all $p \in S$ such that one of the four quadrants of the Cartesian axes centered at p contains no point in S (see [3] for more details). A set's outer layer always contains the set's convex hull. We define $OL(S)$ to be the outer layer points of S and $OL_n^C = |OL(S)|$ to be the number of such points.

2 Results

In what follows we will always assume that C is a *closed* region. We do this to ensure that C contains its boundary, ∂C : this assumption makes our proofs slightly simpler. Notice though that the assumption is not restrictive. If C is *any* bounded convex region then $\mathbf{E}\left(M_n^C\right) = E\left(M_n^{\overline{C}}\right)$ because a point chosen from the uniform distribution over $\overline{C}$ is in ∂C with probability zero. It thus suffices to analyze $\mathbf{E}\left(M_n^C\right)$ for closed C. Our first theorem is

Theorem 1 (The Gap Theorem) *Let C be a planar convex region. We say that a point $p \in C$ is an* upper-right-hand-corner *of C if p dominates every point $q \in C$. The expected number of maxima among n points chosen I.I.D. uniformly from C is qualitatively dependent upon whether C has an upper-right-hand-corner:*

- *If C does not have such a corner then $\mathbf{E}\left(M_n^C\right) = \Theta(\sqrt{n})$.*
- *If C does have such a corner then $\mathbf{E}\left(M_n^C\right) = O(\log n)$.*

Note that for all convex C, $\mathbf{E}\left(M_n^C\right)$ can not behave like a function asymptotically between $\log n$ and $\sqrt{n}$. Hence the name Gap Theorem, alluding to the gap between the two possible behaviours.

Example 1 Figures 1 (b), (c), (d), (f), and (h) all have upper-right-hand-corners and thus have $\mathbf{E}\left(M_n^C\right) = O(\log n)$: figures 1 (a), (e), and (g) don't and so have $\mathbf{E}\left(M_n^C\right) = \Theta(\sqrt{n})$.

Recall that OL_n^C is the number of outer-layer points among n points chosen I.I.D. uniformly from C. Theorem 1 can be used to say quite a lot about the expected number of outer layer points.

Corollary 1 *Let C be a planar convex region. If C is a rectangle with sides parallel to the Cartesian axes then $\mathbf{E}\left(OL_n^C\right) = \Theta(\log n)$. Otherwise $\mathbf{E}\left(OL_n^C\right) = \Theta(\sqrt{n})$.*

Example 2 In Figure 1, (c) has $OL_n^C = \Theta(\log n)$ while all of the other regions have $OL_n^C = \Theta(\sqrt{n})$.

Theorem 1 tells us that when C does not have an upper-right-hand-corner then $\mathbf{E}\left(M_n^C\right) = \Theta(\sqrt{n})$. When C does have such a corner then all that we know is that $\mathbf{E}\left(M_n^C\right) = O(\log n)$. To derive tighter bounds it is necessary to have better information about the tangents to the boundary of C at the corner. We digress momentarily to introduce notation describing these tangents.

Let C be a convex region. If C has an upper-right-hand-corner p then the boundary curve of C as it leaves p can be divided into two parts: one curve that goes down and the other that goes to the left. We define two functions $d(\alpha)$ and $l(\alpha)$:

$$\begin{aligned} d(\alpha) &= \beta \text{ such that } (p.x - \beta, p.y - \alpha) \text{ is on the down curve,} \\ l(\alpha) &= \beta \text{ such that } (p.x - \alpha, p.y - \beta) \text{ is on the left curve.} \end{aligned}$$

See Figure 2. Although these functions are not defined for all real α the convexity of C ensures that there is always some $\epsilon > 0$ such that both functions are well defined, convex and nondecreasing in $[0, \epsilon]$ with $d(0) = l(0) = 0$. Since the functions are convex their left and right derivatives exist everywhere in the interval except for the undefined left derivative at 0 and the undefined right one at ϵ.

The down tangent to C at p is the tangent to the down curve at p. The slope of this tangent line is totally determined by the value of the right derivative $d'_+(0)$. If $d'_+(0) = 0$ the tangent line is vertical. Similarly, if $l'_+(0) = 0$ the left tangent line, the tangent to the left curve, is horizontal. This is illustrated in Figure 2.

The next two theorems discuss the behaviour of $\mathbf{E}\left(M_n^C\right)$ when C has an upper-right-hand-corner .

Theorem 2 *Let C be a convex planar region with an upper-right-hand-corner p. If the down tangent at p is not vertical and the left tangent at p is not horizontal then $\mathbf{E}\left(M_n^C\right) = \Theta(1)$. Otherwise $\mathbf{E}\left(M_n^C\right) = \Omega(1)$.*

Example 3 Figures 1 (b) and (f) have $\mathbf{E}\left(M_n^C\right) = \Theta(1)$; figures 1 (c), (d) and (h) have $\mathbf{E}\left(M_n^C\right) = \Omega(1)$.

We now present a tight lower bound for many of the cases in which the left tangent is horizontal and/or the down tangent is vertical.

Theorem 3 *Let C be a convex planar region with an upper-right-hand-corner p. Suppose further that at least one of the two tangents to C at p fulfills the following (Lipschitz-like) conditions:*

1. *The down tangent is not vertical and there are positive constants δ and c such that $l(\alpha) \leq c\alpha^{1+\delta}$.*
2. *The left tangent is not horizontal and there are positive constants δ and c such that $d(\alpha) \leq c\alpha^{1+\delta}$.*
3. *There are positive constants δ and c such that $d(\alpha) \leq c\alpha^{1+\delta}$ and $l(\alpha) \leq c\alpha^{1+\delta}$.*

Then $\mathbf{E}\left(M_n^C\right) = \Theta(\log n)$.

Condition (1) forces the left tangent to be horizontal and condition (2) forces the down tangent to be vertical. The conditions of the theorem can be thought of as requiring not only the left (down) tangent to be horizontal (vertical) but the curve leaving C itself to be "almost" horizontal (vertical) near p. These conditions might seem artificial but in practice are satisfied quite often. For example:

Example 4 As an application of Theorem 3 we find that if C is a convex *polygon* with an upper-right-hand-corner and a vertical down tangent and/or a horizontal left tangent at the corner then $\mathbf{E}\left(M_n^C\right) = \Theta(\log n)$, e.g. Figures 1 (c) and (h) have $\mathbf{E}\left(M_n^C\right) = \Theta(\log n)$.

Combining this with Theorems 1 and 2 we find that if C is a convex polygon then $\mathbf{E}\left(M_n^C\right)$ can have only one of three possible behaviours: $\Theta(\sqrt{n})$, $\Theta(\log n)$, or $\Theta(1)$.

3 Dual Results

Let C be a planar region. Choose n points $p_1, \ldots, p_n$ I.I.D. from the uniform distribution over C. Let M_n^C be the number of these points that are maximal. If C is convex it is known that either $\mathbf{E}\left(M_n^C\right) = \Theta(\sqrt{n})$ or $\mathbf{E}\left(M_n^C\right) = O(\log n)$. We will show that, for general C, there is very little that can be said, a priori, about $\mathbf{E}\left(M_n^C\right)$. More specifically we will show that if g is a member of a large class of monotonic functions then there is a region C such that $\mathbf{E}\left(M_n^C\right) = \Theta(g(n))$. This class contains all functions with regular variation and (i) exponent less than 1 or (ii) exponent equal to 1 and $n/g(n) > \ln^\beta n$ for some $\beta > 1$. For example, all functions of the form $g(n) = n^\alpha$, $0 < \alpha < 1$, or $g(n) = \ln^\beta n$, $\beta \geq 0$ satisfy condition (i) while all functions of the form $g(n) = n \ln\ln n / \ln^\beta n$, $\beta > 1$ satisfy (ii). The class also contains nondecreasing functions like $g(n) = \ln^* n$. The results in this paper remain valid in higher dimensions.

3.1 Finding a planar region corresponding to "good" monotonic functions

Definition 1 *A positive (not necessarily) monotone function L defined on* $(0, \infty)$ is slowly varying *at infinity if and only if, for all $x > 0$*

$$\lim_{t\to\infty} \frac{L(xt)}{L(t)} \to 1.$$

Definition 2 *A function U defined on* $(0, \infty)$ is regularly varying *at infinity with exponent ρ if and only if it is of the form $x^\rho L(x)$ where L is slowly varying.*

As an example the function $\ln^2 n$ varies slowly at infinity so the function $\sqrt{n}\ln^2 n$ varies regularly at infinity with exponent $1/2$. Similarly, the function $1/\ln^2 n$ varies slowly at infinity so the function $n/\ln^2 n$ varies regularly at infinity with exponent 1.
We now state our main result.

Theorem 4 *Let g be a continuous, monotonically increasing almost everywhere differentiable function from $(0, \infty)$ onto itself. Furthermore, suppose that g is regularly varying with exponent ρ and either (i) $\rho < 1$ or (ii) $\rho = 1$ and $x/g(x) \geq \ln^\beta x$ for some $\beta > 1$. Then there is some planar region C such that for n points $p_1, \ldots, p_n$ chosen I.I.D. uniformly from C the expected number of the points that are maximal is $\Theta(g(n))$:*

$$\mathbf{E}\left(\left|\mathrm{MAX}\left(\{p_1, \ldots, p_n\}\right)\right|\right) = \Theta(g(n)).$$

Very large classes of functions g satisfy the conditions of Theorem 4. Some examples:

1. $g(x) = x^\alpha$ where $\alpha < 1$.
2. More generally, $g(x) = x^\alpha e^{\ln^\beta x} \ln^\gamma x$ where $0 \leq \alpha < 1$, $0 \leq \beta < 1$ and $\gamma > 0$.
3. $g(x) = \ln^{(m)}(x)$ the m'th iterated logarithm: $\ln^{(0)}(x) = x$, $\ln^{(m+1)}(x) = \ln\left(\ln^{(m)}(x)\right)$.
4. $g(x) = \frac{x \ln\ln x}{\ln^\beta x}$ where $\beta > 1$.

Examples 1, 2, and 3 satisfy condition (i); example 4 satisfies condition (ii). The theorem therefore tells us that for each of these g-s there is some C such that $\mathbf{E}\left(M_n^C\right) = \Theta(g(n))$.
A theorem due to Feller [5, page 281], giving the asymptotics of the truncated moments of regularly varying functions, allows us to show that if g satisfies the conditions of Theorem 4 then

$$\sum_{i>g(n)} \frac{1}{g^{-1}(i)} = O\left(\frac{g(n)}{n}\right). \tag{1}$$

We demonstrate then the following theorem:

Theorem 5 *Let g be a continuous monotonically increasing function from $(0,\infty)$ into itself with $g(x) \le x$ and $\lim_{x\to\infty} g(x) = \infty$. Use g^{-1} to denote the functional inverse of g: $g(g^{-1}(x)) = g^{-1}(g(x)) = x$. Suppose that g fulfills the following condition for all integers $n > 0$:*

$$\sum_{i>g(n)} \frac{1}{g^{-1}(i)} = O\left(\frac{g(n)}{n}\right). \tag{2}$$

Then there exists a connected planar region C such that, for n points $p_1, \ldots, p_n$ chosen I.I.D. uniformly from C, the expected number of the points that are maximal is $\Theta(g(n))$:

$$\mathbf{E}\left(|\mathrm{MAX}\left(\{p_1, \ldots, p_n\}\right)|\right) = \Theta(g(n)).$$

The demonstration uses the following lemma:

Lemma 1 *Fix $d > 0$ and let T be the triangle (Figure 3) with vertices $(0,0)$, $(d,0)$ and $(2d,2d)$. Choose n points $p_1, \ldots, p_n$ I.I.D. uniformly from T. Then*

$$\mathbf{E}\left(|\mathrm{MAX}\left(\{p_1, \ldots, p_n\}\right)|\right) \le 2.$$

The demonstration proceeds in two parts, with the construction of two regions C and C'.

Part 1: We construct an infinite sequence C_i of smaller and smaller triangles of the type described by the lemma (Figure 4) and define $C = \bigcup_i C_i$. We show then that when n points are chosen I.I.D. uniformly from $C = \bigcup_i C_i$ then $\mathbf{E}\left(M_n^C\right) = \Theta(g(n))$.

Part 2: We modify C to yield a *connected* region C' such that $\mathbf{E}\left(M_n^{C'}\right) = \Theta(g(n))$. It is this C' that will satisfy the theorem. We need the following lemma:

Lemma 2 *Let $d > 0$ and $d' \le 4d$. Define T to be, as in Lemma 1, the triangle with vertices $(0,0)$, $(d,0)$ and $(2d,2d)$. Let R be the rectangle with vertices $(0,0)$, $(d,0)$, $(0,-d')$ and $(d,-d')$. If $p_1, \ldots, p_n$ are chosen I.I.D. uniformly from $T \cup R$ then*

$$\mathbf{E}(|\mathrm{MAX}(\{p_1, \ldots, p_n\})|) = O(1).$$

The new region C' (Figure 5) will be the union of an infinite number of regions of the type defined by the lemma.

Construction of C' :

1. Set $f_i = 1/\sqrt{g^{-1}(i)}$.
2. Set $x_1 = y_1 = 0$ and for $i > 1$ set $x_i = x_{i-1} + 2f_{i-1}$ and $y_i = y_{i-1} - 2f_i$.
3. Define C'_1 to be the triangle with vertices $(0,0), (f_1, 0)$ and $(2f_1, 2f_1)$. For $i > 1$ define C'_i to be the triangle with vertices $(x_i+2f_i, y_i+2f_i), (x_i-2f_{i-1}, y_i-2f_{i-1})$ and $(x_i+f_i-f_{i-1}, y_i-2f_{i-1})$.
4. For $i > 0$ define R_i to be the rectangle with vertices $(x_i, y_i), (x_i+f_i, y_i), (x_i+f_i, y_i-2f_i-2f_{i+1})$ and $(x_i, y_i - 2f_i - 2f_{i+1})$
5. Set $C' = \bigcup_i (C'_i \cup R_i)$.

3.2 Higher Moments

The regions C constructed in this section were carefully tailored so that $\mathbf{E}\left(M_n^C\right) = \Theta(g(n))$ for given monotonically increasing functions g. There is a rather remarkable theorem due to Devroye [2] which, for $p > 1$, gives us the higher moments $\mathbf{E}\left(\left(M_n^C\right)^p\right)$. This theorem (more specifically the remark 3 following the theorem) states that if $\mathbf{E}\left(M_n^C\right) = \Theta(g(n))$ where g is nondecreasing then $\mathbf{E}\left(\left(M_n^C\right)^p\right) = \Theta\left(\left(\mathbf{E}\left(M_n^C\right)\right)^p\right) = \Theta\left(g^p(n)\right)$. Thus we know all of the higher moments of M_n^C. As an example suppose C was constructed so that $\mathbf{E}\left(M_n^C\right) = \Theta\left(n^{1/3}\right)$. Then $\mathbf{E}\left(\left(M_n^C\right)^p\right) = \Theta\left(n^{p/3}\right)$ for all $p \geq 1$.

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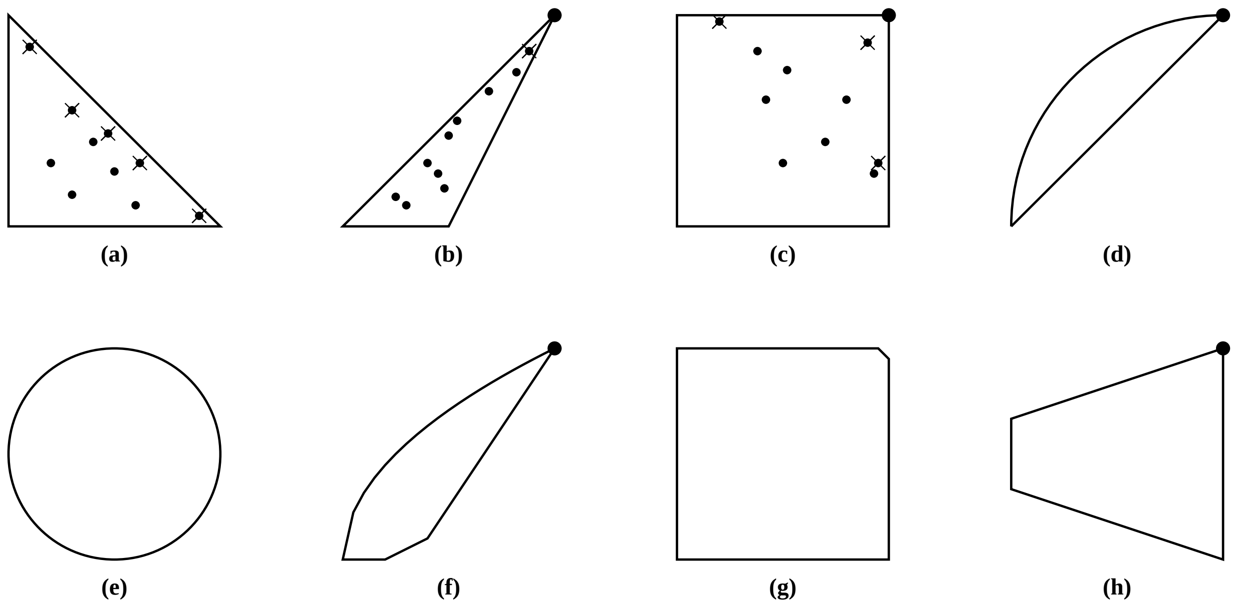

Figure 1: Figures (a), (b), and (c) each contain 10 points the maxima of which are marked by x-s; (a) contains 5 maximal points, (b) 1 maximal point and (c) 3 maximal points. Figures (b), (c), (d), (f) and (h) all have upper-right-hand-corners (marked with a large point); the other figures don't have a corner. Figures (c) and (d) have horizontal left tangents; figures (c) and (h) have vertical down ones.

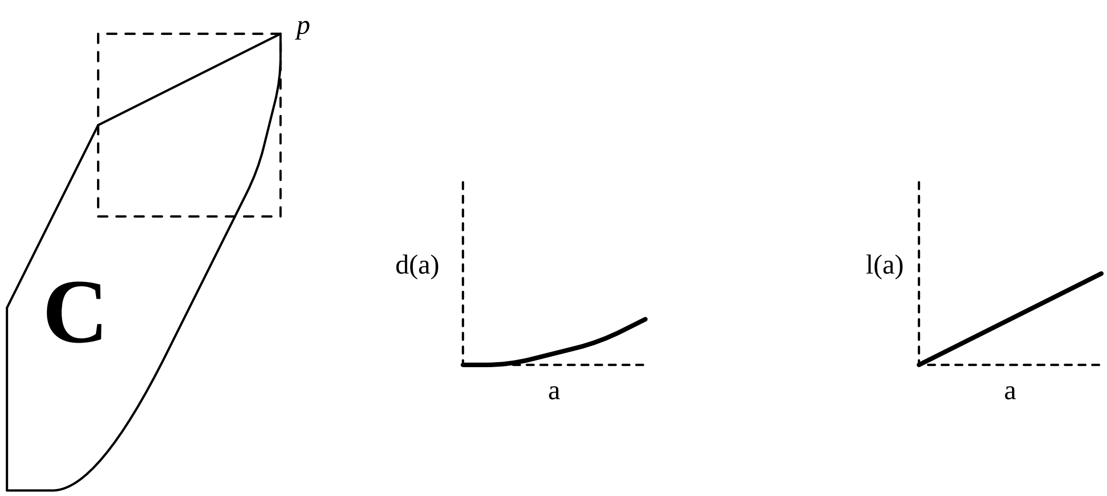

Figure 2: C has an upper-right-hand-corner p with a vertical down tangent at p and a non-horizontal left one. The middle figure portrays $d(\alpha)$, the displacement of the down boundary curve from the vertical line through p and the rightmost figure portrays $l(\alpha)$, the displacement between the left boundary curve and the horizontal line through p.

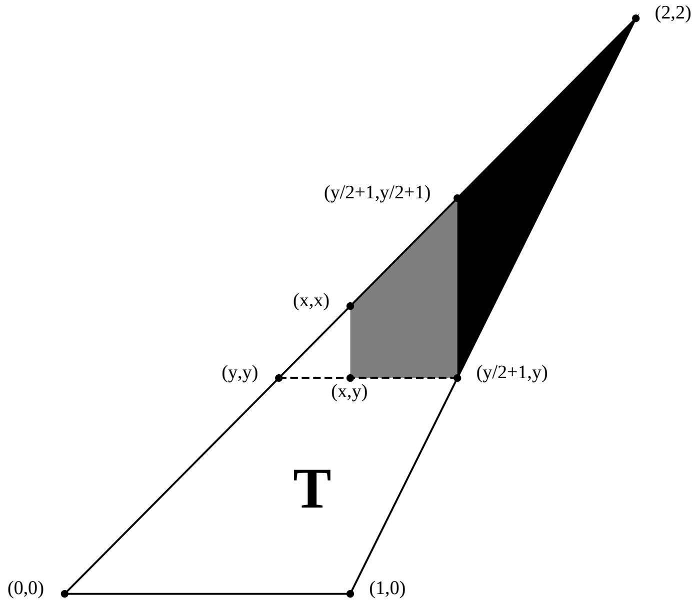

Figure 3: $R(x,y)$ is the region containing all points in T that dominate (x,y). In the diagram $R(x,y)$ is the union of the two shaded regions. The darker of the two shaded regions (the small triangle) has area $\frac{1}{2}\left[y/2+1-y\right]\left[2-(y/2+1)\right] = \frac{1}{2}\left[1-y/2\right]^2$ so Area$(R(x,y)) \geq \frac{1}{2}\left[1-y/2\right]^2$.

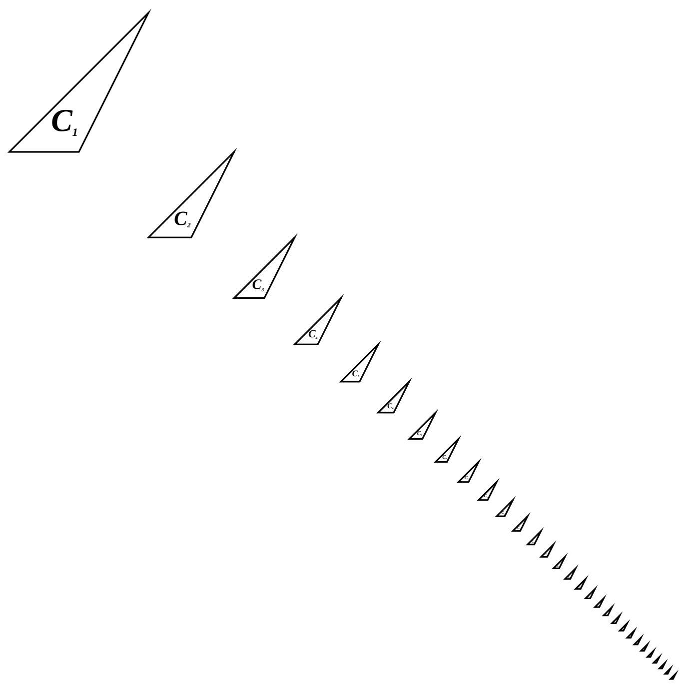

Figure 4: The region $C = \cup_i C_i$ when $g(x) = x^{5/12}$, $g^{-1}(x) = x^{12/5}$ and $f_i = i^{-6/5}$. Note that points in different triangles are incomparable; if $p \in C_i$ and $q \in C_j$ where $i \neq j$ then p and q are incomparable.

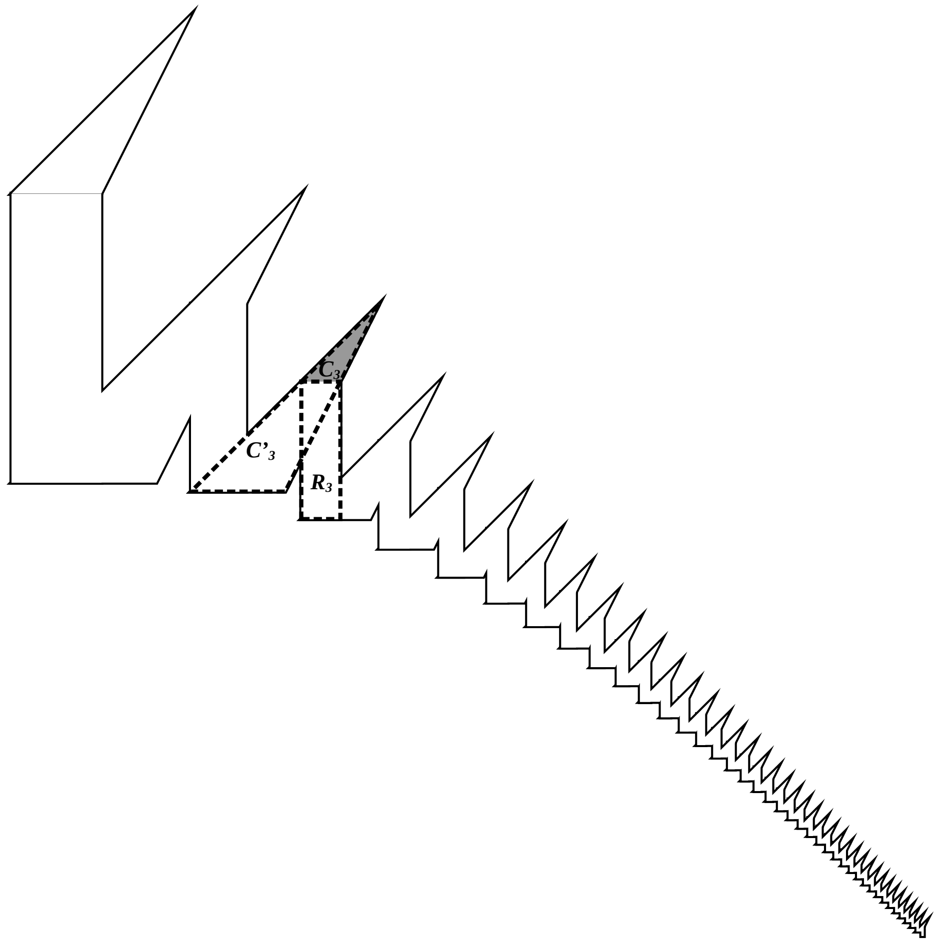

Figure 5: The region $C' = \bigcup_i (C'_i \cup R_i)$ when $g(x) = x^{5/12}$. We have emphasized C'_3 and R_3 by giving them a dashed boundary. The shaded region is the triangle C_3 of the preceding figure. Note how $C_3 \subseteq C'_3$.