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The Asymptotic Behaviour of Coefficients of Large Powers of Functions

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[summary by Bruno Salvy]

The evaluation of the distribution of a sum of independent random variables, the study of asymptotic parameters of trees and forests are two examples of problems that require an asymptotic estimate of the behaviour of $[z^n]f^d(z)$ when both n and d tend to infinity. In this talk, D. Gardy surveys the literature and proves several new results.

The general philosophy in this domain is that the larger d is with respect to n, the easier it is to compute an asymptotic estimate. For instance, if n is fixed and d tends to infinity then elementary majorisations show that $[z^n]f^d(z) \sim [z^n](f_{\alpha_0}z^{\alpha_0} + f_{\alpha_1}z^{\alpha_1})^d$, where f_{α_0} and f_{α_1} are the first non-zero coefficients of f. On the other end of the scale, the problem with fixed d and n tending to infinity is the general problem of asymptotic estimates of coefficients of generating functions which has received a lot of attention, and for which results are known in particular classes. The emphasis in this talk is on the intermediate case, when both n and d tend to infinity.

1 The saddle-point method

The saddle-point method is the method of choice in this domain. We describe it in the general case where it used to get an estimate of $[z^n]\phi(z)$, and will then comment on the implications of taking $\phi(z) = f^d(z)$ with d tending to infinity.

One starts from Cauchy's theorem

$$[z^n]\phi(z) = \frac{1}{2i\pi}\oint \frac{\phi(z)}{z^{n+1}}\,dz,$$

where the contour contains the origin and no other singularity. A suitable contour is a circle through a particular point called *saddle-point*, which is a root of the derivative of the integrand (see [10] for a good intuitive justification of this):

$$\frac{\phi'(\rho)}{\phi(\rho)} = \frac{n+1}{\rho}. \tag{1}$$

If everything goes well, there is such a point on the positive real axis, it is a maximum of the integrand on the circle, locally the integrand behaves as a Gaussian and the other parts of the integral are negligible. If ρ is the only maximum of f on the circle of radius ρ, $h(z) = \log\phi(z) - (n+1)\log z$ and $h''(\rho) \neq 0$ (sufficient conditions for this will be given below), then one can see that $h(z) = h(\rho) + h''(\rho)u^2/2$ defines two functions $u(z)$ analytic at $z = \rho$ that do not vanish on the circle. By imposing further $u'(\rho) > 0$, there is only one such function and we can change the variable in the integral, thus obtaining

$$[z^n]\phi(z) = \frac{\phi(\rho)}{2i\pi\rho^{n+1}} \int e^{h''(\rho)u^2/2} z'(u)\, du.$$

Expanding $z'(u)$ as a power series and integrating term by term from $-\infty$ to ∞ gives a full asymptotic expansion:

$$[z^n]\phi(z) \approx \frac{\phi(\rho)}{\rho^{n+1}\sqrt{2\pi h''(\rho)}} \left[1 + \frac{\alpha_1}{n} + \frac{\alpha_2}{n^2} + \cdots\right].$$

The computations needed by the saddle-point method are technical but not hard. The difficulty lies in the justification of its use. If one only wants the first order saddle-point estimate, obtaining this validity requires three steps: *i)* proving the existence of the saddle-point in a valid part of the complex plane, *ii)* proving that the expansion of $z'(u)$ is valid in a sufficiently large neighbourhood of the saddle-point, *iii)* bounding h on the rest of the contour. A sufficient condition for *ii)* and *iii)* is that there exists a real positive function $\epsilon(n) < 1$ such that

(a.) $|\epsilon^2\rho^2 h''(\rho)| \to \infty, \quad n \to \infty;$

(b.) $|h^{(3)}(\rho e^{i\theta})| = o\left(\frac{h''(\rho)}{\rho\epsilon}\right)$, uniformly for $0 \le |\theta| \le \epsilon$.

(c.) $|\phi(\rho e^{i\theta})| = o\left(\frac{\phi(\rho)}{\rho\sqrt{h''(\rho)}}\right)$, uniformly for $\pi > |\theta| > \epsilon$.

2 Known results

Considering a generating function ϕ with non negative real Taylor coefficients at the origin which tends to infinity "rapidly enough" at its finite or infinite singularity, it is not difficult to show that the saddle-point equation (1) has roots. Using the triangular inequality, one gets that one of the saddle-points of smallest modulus is a real positive number, and that its modulus is less than the radius of convergence of ϕ. If, besides, the gcd of the indices of the non-zero coefficients of ϕ is 1, then this saddle-point is the only one with this modulus.
What makes it hard to get a majorisation of ϕ on the circle of radius ρ (condition *(c.)* above) is that when n tends to infinity, this circle can get close to singularities and other saddle-points of the function. This happens for instance with $\phi(z) = z + e^{z^2}$.
When d is fixed, this problem is solved for large classes of functions (called "admissible") either by requiring the functions to satisfy more or less stringent conditions (see [6, 7, 9]), or by taking into account the other contributions to the integral [12].
The effect of d tending to infinity when $\phi = f^d$ is *i)* to increase the difference between the largest value of f on the contour and the other ones, *ii)* to make ϕ grow fast enough; both properties make it easier to prove the validity of the saddle-point method. Thus with the following hypotheses:

H1: f has non negative real coefficients and $\gcd\{k \mid [z^k]f(z) \neq 0\} = 1$;

H2: $0 < a < d/n < b$ for some real a and b (depending on f);

Daniels has proved in [2] that the saddle-point method applies. This result is reformulated and slightly improved (without proof) in [5][5].
What makes hypothesis **H1** very useful is the following lemma.

Lemma 1 *Let f be a generating function with radius of convergence $R \leq \infty$. If f satisfies* **H1**, *then for any $r < R$, $\theta \mapsto |f(re^{i\theta})|$ has a single maximum in $(-\pi, \pi)$, which is attained for $\theta = 0$.*

This was proved by Daniels and probably many others before.

In another context, Meir and Moon have proved in [8] the applicability of the saddle-point method to the coefficients of f^d when f is solution to $f(x) = x\phi(f(x))$, where ϕ satisfies **H1** and $\phi(0) = 1$, with the following restriction on d: $d = \alpha n + \lambda\sqrt{n} + o(n^{1/2})$, α being possibly zero. Intuitively, the reason for this is that the saddle-point method already applies when $d = 1$ (see [11]).

3 The case $n = o(d)$

This is the case studied by D. Gardy in this talk, following [3]. Then the smallest solution of (1) obviously tends to zero, and one can even give an asymptotic estimate for ρ. All the computations then get simpler and particularly the determination of a domain in which the function is "sufficiently" Gaussian. D. Gardy's main theorem is the following.

Theorem 1 *Let f satisfy* **H1** *and $n = o(d)$. Then (1) has a unique real positive root ρ, and asymptotically*

$$[z^n]f^d(z) = \frac{f^d(\rho)}{\rho^n\sqrt{2\pi n}}(1 + o(1)), \quad n \to \infty.$$

The proof strategy is as follows:

1. first prove the existence of ρ by an intermediate value argument and compute an asymptotic estimate by inversion:
$$\rho = \frac{f(0)}{f'(0)}\frac{n}{d}(1 + O(n/d));$$
2. from this deduce an estimate for $f^{(k)}(\rho) = f^{(k)}(0) + O(n/d)$ and
$$\rho^2 h''(\rho) = n(1 + O(n/d) + O(1/n)), \quad \frac{\rho h^{(3)}(\rho e^{i\theta})}{h''(\rho)} = -2 + O(n/d) + O(\theta);$$
3. these estimates show that the function $\epsilon(n) = n^{-1+1/k}$ for any integer $k > 1$ satisfies conditions *(a.)* and *(b.)*, and a few extra computations with the help of Lemma 1 show that *(c.)* is also fulfilled.

D. Gardy then proceeds with the study of $[z^n]f^d(z)\psi(z)$ with the same hypothesis on f and d and a function ψ analytic and non-zero at the origin, to be specified. In this case, it is known that the saddle-point method still works with the same saddle-point and the same function $\epsilon(n)$ as before provided that

[5] As it is stated, Good's theorem (p.868) is wrong, a counter-example being $[z^n](1+z)^n$.

(d.) $|\psi(\rho e^{i\theta})| \sim \psi(\rho)$ uniformly for $0 \leq |\theta| \leq \epsilon$ or;

(d'.) uniformly for $0 \leq |\theta| \leq \epsilon$, the following estimates hold:

$$\left|\frac{\psi'}{\psi}\right|(\rho) = o(\sqrt{h''(\rho)}), \qquad \left|\left(\frac{\psi'}{\psi}\right)'\right|(\rho e^{i\theta}) = o(h''(\rho e^{i\theta}));$$

(e.) $|\psi(\rho e^{i\theta})| = O(\psi(\rho))$ uniformly for $\epsilon < |\theta| < \pi$;

and then the asymptotic estimate has to be multiplied by $\psi(\rho)$. Conditions *(d.)* and *(e.)* are obviously fulfilled if ψ is an analytic function which does not depend on d or n, but more generally, D. Gardy considers functions ψ satisfying **H1** and of the form

$$\psi(z) = \prod_{i=1}^{p} g_i(z)^{d_i}, \quad d_i = o(d/\sqrt{n}).$$

Since $\psi(0) \neq 0$, it is easy to see that *(d'.)* is then satisfied, and *(e.)* follows immediately from Lemma 1.

Note. The sufficient conditions used in this summary are taken or derived from [1].

References

[1] L. Berg. Asymptotics of integrals, series, and operators. In R. Wong, editor, *Asymptotic and Computational Analysis*, volume 124 of *Lecture notes in pure and applied mathematics*, pages 35–52. M. Dekker, New York, Basel, 1990. Conference in Honor of Frank W. J. Olver's 65th Birthday.

[2] H. E. Daniels. Saddlepoint approximations in statistics. *Annals of Mathematical Statistics*, 25:631–650, 1954.

[3] D. Gardy. On coefficients of powers of functions. Technical Report CS-91-53, Brown University, August 1991.

[4] D. Gardy. Some results on the asymptotic bahaviour of coefficients of large powers of functions. Rapport de recherche 769, Laboratoire de recherche en informatique, Université de Paris Sud, August 1992.

[5] I. J. Good. Saddle-point methods for the multinomial distribution. *Annals of Mathematical Statistics*, 28:861–881, 1957.

[6] B. Harris and L. Schoenfeld. Asymptotic expansions for the coefficients of analytic functions. *Illinois Journal of Mathematics*, 12:264–277, 1968.

[7] W. K. Hayman. Coefficient problems for univalent functions and related function classes. *Journal of the London Mathematical Society*, 40:385–406, 1965.

[8] A. Meir and J. W. Moon. The asymptotic behaviour of coefficients of powers of certain generating functions. *European Journal of Combinatorics*, 11:581–587, 1990.

[9] A. M. Odlyzko and L. B. Richmond. Asymptotic expansions for the coefficients of analytic generating functions. *Aequationes Mathematicae*, 28:50–63, 1985.

[10] L. Sirovich. *Techniques of Asymptotic Analysis*. Springer Verlag, 1971.

[11] J.-M. Steyaert and Ph. Flajolet. Patterns and pattern-matching in trees: an analysis. *Information and Control*, 58(1–3):19–58, July 1983.

[12] M. Wyman. The asymptotic behavior of the Laurent coefficients. *Canadian Journal of Mathematics*, 11:534–555, 1959.