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Counting Convex Polyominoes According to Their Area

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[summary by Dominique Gouyou-Beauchamps]

Two approaches are presented for the enumeration according area of different classes of convex polyominoes. The first approach is based on the concept of coins stacking. In the second approach, the elementary decomposition of polyominoes leads, for each class, to a q-equation, that is sometimes solvable.

Unit squares with vertices at integer points in the cartesian plane are called *cells*. A *polyomino* is a finite connected union of cells such that the interior is also connected (no cut point). The *area* of a polyomino is the number of cells, the *perimeter* is the length of the border. Polyominoes are defined up to a translation. A *column* (resp. *row*) of a polyomino is the intersection between the polyomino and any infinite vertical (resp. horizontal) strip of unit squares. A polyomino is said to be *column-* (resp. *row-*) *convex* if all its columns (resp. rows) are connected. A *convex* polyomino is both row- and column-convex.

Let P a convex polyomino and $Rect(P)$ be the smallest *rectangle* (considered as a convex polyomino) containing P. The polyomino touches the border of $Rect(P)$ along four connected segments. Each of these segments has two extreme points and thus we introduce 8 points, as shown in Figure 1. The Westmost (respectively Eastmost) of the points of P containing the South (respectively North) border of $Rect(P)$ is denoted by $S(P)$ (respectively $N(P)$). Following counterclockwise the border of P, one meets successively the above 8 canonical points in the order: $S(P)$, $S'(P)$, $E(P)$, $E'(P)$, $N(P)$, $N'(P)$, $W(P)$, $W'(P)$. The *height* and the *length* of the convex polyomino P are the height and the length of the rectangle $Rect(P)$ (see Figure 1). We can now define several important subclasses of convex polyominoes. A *parallelogram polyomino* is a convex polyomino P such that $S(P) = W'(P)$ and $N(P) = E'(P)$ (see Figure 2). A *stack polyomino* is a convex polyomino such that $S(P) = W'(P)$ and $S'(P) = E(P)$ (see Figure 2). A *Ferrers diagram* is a convex polyomino P such that $N(P) = E'(P)$, $S'(P) = E(P)$ and $S(P) = W'(P)$ (see Figure 2). A *directed convex polyomino* is a convex polyomino P such that $N(P) = E'(P)$ (see Figure 2). Polyominoes are very classical objects in combinatorics. Counting polyominoes according to their area or perimeter is a major unsolved problem in combinatorics. The first authors interested in this subject were Read [8] and Golomb [7]. Physicists have given several asymptotic results. They call *animal* a set of points obtained by taking the centers of the cells of a polyomino. Giving a privileged direction for the growth of an animal allows them to obtain generating functions (see Viennot [9] and references therein). In theoretical computer science, Yuba and Hoshi [11] introduced directed polyominoes (or animals) for a new method for key searching. They consider that a polyomino is a *binary search network* structure.

The following sections give some enumeration results for different classes of polyominoes.

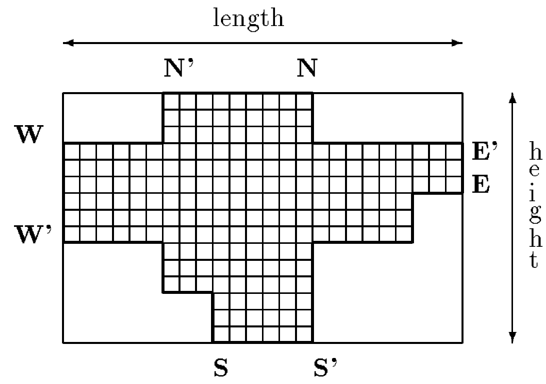

Figure 1: A convex polyomino

1 Counting convex polyominoes according to their length and their height

Theorem 1 *The generating function for the number $p_{m,n}$ of convex polyominoes having height m and length n is:*

- $\sum_{m,n} p_{m,n} x^m y^n = \frac{xy}{1-x-y}$ *for Ferrers diagrams,*
- $\sum_{m,n} p_{m,n} x^m y^n = \frac{xy(1-x)}{1-2x-y+x^2}$ *for stack polyominoes,*
- $\sum_{m,n} p_{m,n} x^m y^n = \frac{1-x-y-\sqrt{\Delta}}{2}$ *for parallelogram polyominoes,*
- $\sum_{m,n} p_{m,n} x^m y^n = \frac{xy}{\sqrt{\Delta}}$ *for directed convex polyominoes (Chang and Lin [3]),*
- $\sum_{m,n} p_{m,n} x^m y^n = \frac{xy}{\Delta^2}(1-3x-3y+3x^2+3y^2+5xy-x^3-y^3-x^2y-xy^2-xy(x-y)^2)-\frac{4x^2y^2}{\Delta^{3/2}}$ *for convex polyominoes (Chang and Lin [3], Delest and Viennot [4]), where* $\Delta = 1-2x-2y-2xy+x^2+y^2$.

2 Counting convex polyominoes according to their length, their height and their area

Theorem 2 *The generating function for the numbers $f_{m,n,a}, s_{m,n,a}$ of convex polyominoes having height m, length n and area a is:*

- $\sum_{m,n,a} f_{m,n,a} x^m y^n q^a = \sum_{m\geq 1} \frac{xy^m q^m}{(yq)_m}$ *for Ferrers diagrams,*
- $\sum_{m,n,a} s_{m,n,a} x^m y^n q^a = \sum_{m\geq 1} \frac{xy^m q^m}{(yq)_{m-1}(yq)_m}$ *for stack polyominoes, where* $(a)_n = (1-a)(1-aq)(1-aq^2)...(1-aq^n)$ *and* $(a)_0 = 1$.

In [5] and [6], Delest and Fédou give the generating function for the number of parallelogram polyominoes according to their length and their area.

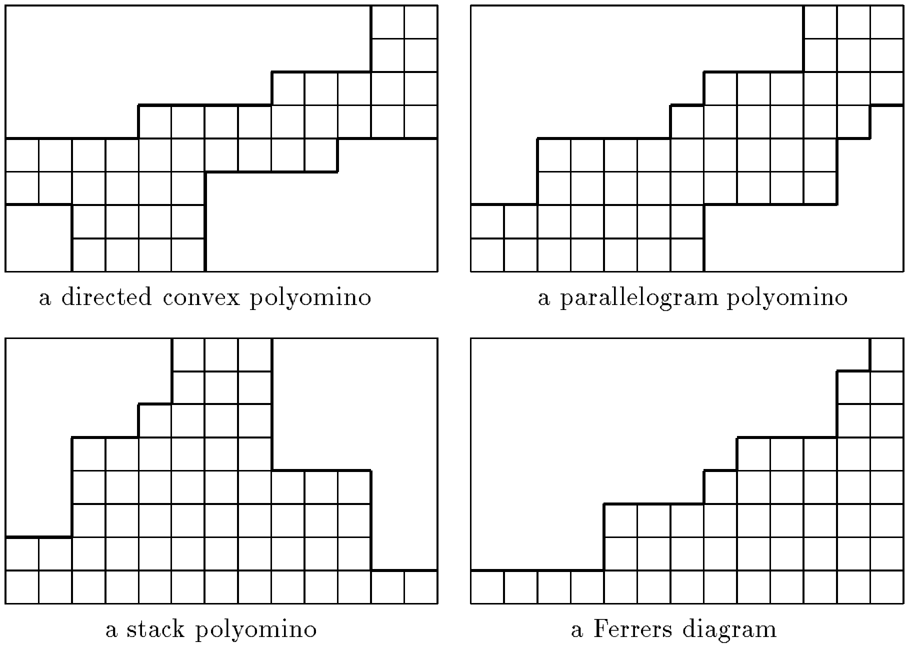

Figure 2: Different classes of convex polyominoes

3 Stack polyominoes

Theorem 3 *The generating function for the number $s_{m,n,a}$ of stack polyominoes having height m, length n and area a is:*

$$\sum_{m,n,a} s_{m,n,a} x^m y^n q^a = \sum_{m\geq 1} \frac{xy^m q^m}{(yq)_{m-1}(yq)_m} = \sum_{n\geq 1} \frac{xy^n q^n T_n}{(xq)_n},$$

where

$$(a)_n = (1-a)(1-aq)(1-aq^2)...(1-aq^n) \text{ and } (a)_0 = 1,$$

$$T_n = 1 + \sum_{2\leq 2k\leq n} x^k q^{k^2} \sum_{m=k}^{n-k} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n-m-1 \\ k-1 \end{bmatrix}_q,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[k]![n-k]!} \text{ and } [n]! = 1(1+q)(1+q+q^2)...(1+q+...+q^{n-1}).$$

4 Parallelogram polyominoes

Theorem 4 *The generating function for the number $p_{m,n,a}$ of parallelogram polyominoes having height m, length n and area a is:*

$$\sum_{m,n,a} p_{m,n,a} x^m y^n q^a = y\frac{N_1}{N} \text{ where } N = \sum_{n\geq 0} \frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_n (yq)_n} \text{ and } N_1 = \sum_{n\geq 1} \frac{(-1)^{n+1} x^n q^{\binom{n+1}{2}}}{(q)_{n-1}(yq)_n},$$

$$(a)_n = (1-a)(1-aq)(1-aq^2)...(1-aq^n),\ n\geq 1 \text{ and } (a)_0 = 1.$$

The proof uses the concept of *heaps of pieces* introduced by Viennot [10] (see also [2]).

Theorem 5 *The generating function for the number $p_{h,m,n,a}$ of parallelogram polyominoes having height m, length n, area a and their first (or last) column of height h is:*

$$\sum_{m,n,a} p_{h,m,n,a} x^m y^n q^a = x y^h q^h \frac{N(xq^h)}{N(x)} \text{ where, as in Theorem 4, } N(x) = \sum_{n\geq 0} \frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_n (yq)_n}.$$

5 Directed convex polyominoes

Theorem 6 *The generating function for the number $d_{m,n,a}$ of directed convex polyominoes having height m, length n and area a is:*

$$\sum_{m,n,a} d_{m,n,a} x^m y^n q^a = y \frac{N_1+N_2}{N} \text{ where } N = \sum_{n\geq 0} \frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_n (yq)_n},\ N_1 = \sum_{n\geq 1} \frac{(-1)^{n+1} x^n q^{\binom{n+1}{2}}}{(q)_{n-1} (yq)_n}$$

$$\text{and } N_2 = y \sum_{n\geq 2} \left(\frac{x^n q^n}{(yq)_n} \sum_{m=0}^{n-2} \frac{(-1)^m x^n q^{\binom{m+2}{2}}}{(q)_m (yq^{m+1})_{n-m-1}} \right).$$

The fundamental idea of the proof is to split a directed convex polyomino into two simpler polyominoes (see Figure 3).

6 Convex polyominoes

Theorem 7 *The generating function for the number $c_{m,n,a}$ of convex polyominoes having height m, length n and area a is:*

$$Z(x,y,q) = \sum_{m,n,a} c_{m,n,a} x^m y^n q^a = 2y \frac{(N_1+N_2)N_3}{N} - 2yZ_1 - Z_2$$

where N, N_1 and N_2 are defined in Theorem 6 and where

$$N_3 = \sum_{m\geq 0} \frac{xq^{m+1} T_m M_{m+1}}{(xq)_m},\ Z_1 = \sum_{0\leq n\leq m} \frac{M_{m+1}^{n+1} T_m T_n}{(xq)_{n-1}(xq)_n},\ Z_2 = \sum_{n\geq 1} \frac{xy^n q^n (T_n)^2}{(xq)_{n-1}(xq)_n}$$

$$T_0 = T_1 = 1,\ T_n = 2T_{n-1} + (xq^{n-1} - 1)T_{n-2},$$

$$M_0 = 0,\ M_1 = 1,\ M_n = (1 + y - xq^{n-1})M_{n-1} - yM_{n-2},$$

$$M_n^n = -xy^{n-1}q^n,\ M_m^n = x^2 y^{n-1} q^{n+m} M_{m-n}(xq^n) \text{ if } m > n.$$

The fundamental idea of the proof is to split a convex polyomino into three simpler polyominoes as in [4]. With a bisection we obtain the following expression for $Z(x,y,q)$:

$$Z(x,y,q) = 2y \sum_{m\geq 1} \frac{y^{m+2}(T_{m+1}S(xq^m) - yT_m S(xq^{m+1}))^2}{[(xq)_m]^2 N(xq^{m-1}) N(xq^m)} + \sum_{m\geq 1} \frac{xy^m q^m (T_m)^2}{(xq)_{m-1}(xq)_m},$$

$$\text{where } S(x) = \sum_{n\geq 1} \left(\frac{x^n q^n}{(yq)_n} \sum_{j=0}^{n-1} \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j (yq^{j+1})_{n-j}} \right).$$

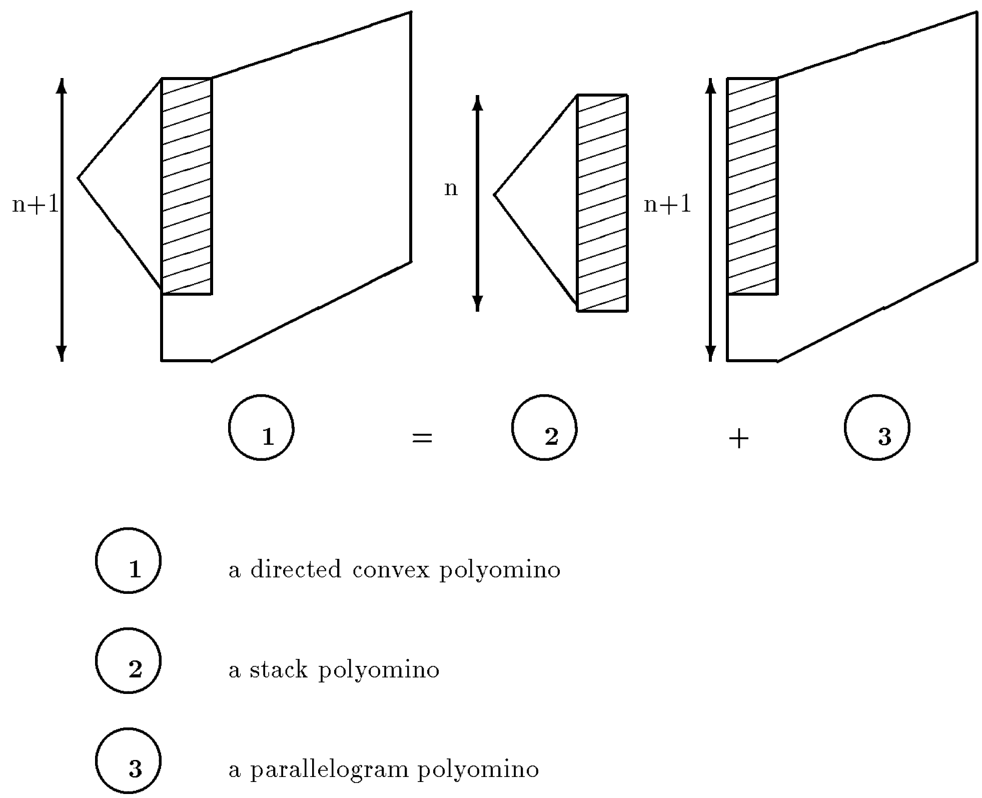

Figure 3: Decomposition of a directed convex polyomino

7 Functional equations

Theorem 8 *The generating function* $P(t) = P(s,t,x,y,q) = \sum_{m,n,a} p_{u,v,m,n,a} s^u t^v x^m y^n q^a$ *of parallelogram polyominoes, having height* m*, length* n*, area* a*, their last column of height* u *and their first column of height* v*, satisfies the following equation:*

$$P(t) = x\frac{styq}{1-styq} + x\frac{tq}{(1-tq)(1-tyq)}(P(1) - P(tq)).$$

This equation can be easily solved. The solution for $s = y = 1$ gives a result of Delest and Fédou [5] (enumeration according to area and number of columns) and for $s = 1$, $x = y = z^2$, gives a result of Guttman (enumeration according to area and perimeter).

Theorem 9 *The generating function* $Y(t) = Y(s,t,x,y,q) = \sum_{m,n,a} y_{u,v,m,n,a} s^u t^v x^m y^n q^a$ *of directed convex polyominoes, having height* m*, length* n*, area* a*, their last column of height* u *and their first column of height* v*, satisfies the following equations:*

$$Y(t) = \frac{xstyq}{1-styq} + tyqT(t) + \frac{xtq}{(1-tq)(1-tyq)}(Y(1) - Y(tq)) \text{ where } T(t) = \sum_{m\geq 2} \frac{x^m q^m yts}{(tyq)_{m-1}(1-tsyq^m)},$$

$$\text{and also } Y(s) = xstyq\frac{1-syq}{1-styq} + syqP(s) + \frac{xsq}{1-sq}Y'(1) + \frac{xs^2q^2}{(1-sq)^2}(Y(sq) - Y(1)).$$

Theorem 10 *The generating function $Z(s) = Z(s,t,x,y,q) = \sum_{m,n,a} z_{u,v,m,n,a} s^u t^v x^m y^n q^a$ of convex polyominoes, having height m, length n, area a, their last column of height u and their first column of height v, satisfies the following equation:*

$$Z(s) = xstyq\frac{1-2syq}{1-styq} + s^2y^2q^2T(t,s) + 2syqY(t,s) + \frac{xsq}{1-sq}Z'(1) + \frac{xs^2q^2}{(1-sq)^2}(Z(sq) - Z(1)).$$

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