

# An analogue of Stokes phenomenon for $q$ -difference equations

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## Abstract

In a common work<sup>1</sup> with Jean-Pierre Ramis and Changgui Zhang, we described an analogue of the Stokes phenomenon for linear analytic complex  $q$ -difference equations and used it to get the local analytic classification. If time permits, I will also show how it was applied in a common work with J.-P. R. the Galois theory of such equations.

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<sup>1</sup>Accepted for publication by Astérisque; meanwhile, see URL

## Origin

The program of analytic classification of  $q$ -difference equations was first proposed and realized by Birkhoff in 1913 in the context of a unified treatment of the Riemann-Hilbert correspondence for *fuchsian* differential, difference and  $q$ -difference equations. The classification program was extended by Birkhoff and Guenter in 1941 for irregular equations, but never pursued:

“Up to the present time, the theory of linear  $q$ -difference equations has lagged noticeably behind the sister theories of linear difference and differential equations. In the opinion of the authors, the use of the canonical system, as formulated above in a special case, is destined to carry the theory of  $q$ -difference equations to a comparable degree of completeness. This program includes in particular *the complete theory of convergence and divergence of formal series, the explicit determination of the essential transcendental invariants (constants in the canonical form), the inverse Riemann theory both for the neighborhood of  $x = \infty$  and in the complete plane (case of rational coefficients), explicit integral representation of the solutions, and finally the definition of  $q$ -sigma periodic matrices, so far defined essentially only in the case  $n = 1$ .* Because of its extensiveness this material cannot be presented here.”

G.D. Birkhoff, 1941

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# Generalities

## General notations

$$q \in \mathbf{C}, |q| > 1.$$

For  $f \in K := \mathbf{C}(\{z\})$  or  $f \in \hat{K} := \mathbf{C}((z))$ :

$$\sigma_q f(z) := f(qz).$$

A (complex analytic) linear  $q$ -difference equation writes:

$$f(q^n z) + a_1(z)f(q^{n-1}z) + \cdots + a_n(z)f(z) = 0,$$

where  $a_1, \dots, a_n \in K$ ,  $a_n \neq 0$ .

Encoding:  $Lf = 0$ ,

$$\text{where } L := \sigma_q^n + a_1 \sigma_q^{n-1} + \cdots + a_n \in \mathcal{D}_{q,K},$$

$$\mathcal{D}_{q,K} := K \langle \sigma_q, \sigma_q^{-1} \rangle, \text{ (Ore ring),}$$

and  $a_1, \dots, a_n \in K$ ,  $a_n \neq 0$ .

**Formal equation:** replace  $K$  by  $\hat{K}$ .

# Generalities

## Equations, systems, $q$ -difference modules

By vectorialisation the  $q$ -difference equation  $Lf = 0$  can be turned into a  $q$ -difference system:

$$\sigma_q X = AX, A \in \mathrm{GL}_n(K), \text{ where } X = \begin{pmatrix} f \\ \vdots \\ \sigma_q^{n-1} f \end{pmatrix},$$

then into a  $q$ -difference module

$$M = (E, \Phi), \text{ with } E := K^n, \Phi := \Phi_A : X \mapsto A^{-1} \sigma_q X.$$

(Compare with vector spaces equipped with a connection.)

Equivalently,  $M$  is a left  $\mathcal{D}_{q,K}$ -module of finite length.

**Formal equation:** replace  $K$  by  $\hat{K}$ .

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Morphisms from  $(K^n, \Phi_A)$  to  $(K^n, \Phi_B)$  correspond to matrices  $F \in \mathrm{GL}_n(K)$  such that  $(\sigma_q F)A = BF$ .

Thus, if  $Y = FX$ , then  $\sigma_q X = AX \Rightarrow \sigma_q Y = BY$ .

**Local analytic classification:** we say that  $A \sim B$  if there exists a gauge transformation  $F \in \mathrm{GL}_n(K)$  such that:

$$B = F[A] := (\sigma_q F)AF^{-1}.$$

**Formal classification:** the same with  $F \in \mathrm{GL}_n(\hat{K})$ .



# Generalities

## Newton polygon (at 0)

The  $q$ -difference operator  $P$  has a *Newton polygon* at 0, which consists in slopes  $\mu_1 < \dots < \mu_k \in \mathbf{Q}$  together with their multiplicities  $r_1, \dots, r_k \in \mathbf{N}^*$ . (Precise definition omitted !)

By the cyclic vector lemma, any  $q$ -difference module can be written  $M = \mathcal{D}_{q,K}/\mathcal{D}_{q,K}P$ .

### Theorem and definition

*The Newton polygon of  $M = \mathcal{D}_{q,K}/\mathcal{D}_{q,K}P$  depends only on the formal isomorphism class of  $M$ .*

### Caution !

By vectorialisation, equation  $L \rightsquigarrow$  system  $A \rightsquigarrow q$ -difference module  $M$ . By the cyclic vector lemma  $M = \mathcal{D}_{q,K}/\mathcal{D}_{q,K}P$ , where  $P$  is “dual” to  $L$ : they have *symetric* Newton polygons and *opposite* slopes.

# Generalities

## Fundamental solutions, constants

One can prove that an analytic system  $\sigma_q X = AX$ ,  
 $A \in \mathrm{GL}_n(K)$  always has a fundamental solution:

$$\mathcal{X} \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}^*, 0)),$$

*i.e. uniform* in a punctured neighborhood of 0.

Therefore, all uniform meromorphic solutions of  $\sigma_q X = AX$   
have the form  $X = \mathcal{X}C$ , where  $C \in (\mathcal{M}(\mathbf{C}^*, 0)^{\sigma_q})^n$ .

The *field of constants*:

$$\mathcal{M}(\mathbf{C}^*, 0)^{\sigma_q} := \{f \in \mathcal{M}(\mathbf{C}^*, 0) \mid \sigma_q f = f\}$$

can be identified with the field of elliptic functions  $\mathcal{M}(\mathbf{E}_q)$ ,

$$\mathbf{E}_q := \mathbf{C}^*/q^{\mathbf{Z}} \simeq \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau), \text{ where } e^{2i\pi\tau} = q.$$

(Identification through the map  $x \mapsto z := e^{2i\pi x}$ .)

# Generalities

## Associated vector bundle

This is for analytic systems (over  $K$ ). One defines:

$$F_A^{(0)} := \frac{(\mathbf{C}^*, 0) \times \mathbf{C}^n}{(z, X) \sim (qz, A(z)X)} \longrightarrow \frac{(\mathbf{C}^*, 0)}{z \sim qz} = \mathbf{E}_q.$$

This is a holomorphic vector bundle over the complex torus (or elliptic curve)  $\mathbf{E}_q$ .

The sheaf of holomorphic solutions of  $\sigma_q X = AX$  near 0 is canonically isomorphic to the sheaf of sections of  $F_A^{(0)}$

$A \rightsquigarrow F_A^{(0)}$  is a “good” functor for classification and for Galois theory (faithful, exact,  $\otimes$ -compatible).

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# Slope filtration

Pure modules, equations, systems

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A module with one slope only is called *pure isoclinic*.

Pure isoclinic modules of slope 0 are *fuchsian* modules. They have the shape  $(K^n, \Phi_A)$ , with  $A \in \mathrm{GL}_n(\mathbf{C})$ . Their analytic and formal classification (due to Birkhoff) are the same.

Pure isoclinic modules of slope  $\mu \in \mathbf{Z}$  have the shape  $(K^n, \Phi_{z^\mu A})$ , with  $A \in \mathrm{GL}_n(\mathbf{C})$ . Their classification boils down to the fuchsian case.

Pure isoclinic modules of nonintegral slope have been classified by van der Put and Reversat in 2005.

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## The canonical filtration

### Theorem

Any  $q$ -difference module over  $K$  admits a unique filtration  $(M_{\leq \mu})_{\mu \in \mathbf{Q}}$  such that each  $M_{(\mu)} := \frac{M_{\leq \mu}}{M_{< \mu}}$  is pure isoclinic of slope  $\mu$ . The filtration is functorial and  $gr : M \rightsquigarrow \bigoplus M_{(\mu)}$  is a faithful exact  $\mathbf{C}$ -linear  $\otimes$ -compatible functor.

### Theorem

Over  $\hat{K}$ , the filtration splits canonically. After formalization (base change  $\hat{K} \otimes_K -$ ),  $gr$  becomes isomorphic to the identity functor.

Note that, contrary to the second, the first theorem has no equivalent in the case of differential equations: it is a consequence of Adams lemma (existence of an analytic factorisation for  $q$ -difference operators).

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## Classification and graduation

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A direct sum of pure isoclinic modules is called *pure*.

### Corollary

*For pure modules, formal and analytic classification are equivalent. Formal classification of an analytic  $q$ -difference module  $M$  amounts to classification (formal or analytic) of the pure module  $\text{gr}M$ .*

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### We already know:

The formal classification, *i.e.* classification of pure  $q$ -difference modules.

### We want to study:

The analytic classification within a formal class, *i.e.* with  $\text{gr}M$  fixed.

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# Irregular equations

## Isoformal classes

The following definition is inspired by Babbitt and Varadarajan “Local moduli for meromorphic differential equations” (Astérisque 169-170).

Fix a pure module  $M_0 := P_1 \oplus \cdots \oplus P_k$ , where  $P_1, \dots, P_k$  are pure isoclinic with slopes  $\mu_1 < \cdots < \mu_k$  and ranks  $r_1, \dots, r_k$ .

Define  $\mathcal{F}(M_0) = \mathcal{F}(P_1, \dots, P_k)$  as the quotient set of pairs  $(M, u)$ , where  $u : \text{gr}M \simeq P_1 \oplus \cdots \oplus P_k$ , up to the equivalence relation:

$$(M, u) \sim (M', u') \iff \exists f : M \rightarrow M' : u = u' \circ \text{gr}f.$$

## Example

Two slopes, one level:

$$\mathcal{F}(P_1, P_2) = \text{Ext}(P_2, P_1).$$

# Irregular equations

The space of analytic classes

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## Theorem

One gets an affine space (actually, a scheme) of dimension:

$$\dim \mathcal{F}(P_1, \dots, P_k) = \sum_{1 \leq i < j \leq k} r_i r_j (\mu_j - \mu_i).$$

(There is a  $q$ -Gevrey version.)

This dimension is equal to the *irregularity* of  $\text{End}(M_0)$ .

From now on,  
the slopes will be assumed to be integral:

$$\mu_1, \dots, \mu_k \in \mathbf{Z}.$$

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## Matricial description

A formal class is encoded by  $M_0 = (K^n, \Phi_{A_0})$ , with:

$$A_0 := \begin{pmatrix} z^{\mu_1} A_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & z^{\mu_k} A_k \end{pmatrix}.$$

An analytic class within  $\mathcal{F}(M_0)$  can then be represented by  $M := (K^n, \Phi_A)$  with:

$$A = A_U := \begin{pmatrix} z^{\mu_1} A_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & U_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & z^{\mu_k} A_k \end{pmatrix},$$

for some  $U := (U_{i,j})_{1 \leq i < j \leq k} \in \prod_{1 \leq i < j \leq k} \text{Mat}_{r_i, r_j}(K)$ .

# Irregular equations

## Birkhoff-Guenther normal form

Using  $q$ -Borel transforms one gets an explicit (algorithmic) computation of *Birkhoff-Guenther normal form*:

### Theorem

Each class in  $\mathcal{F}(P_1, \dots, P_k)$  admits a unique representative  $(K^n, \Phi_{A_U})$  such that each block  $U_{i,j}, 1 \leq i < j \leq k$  has coefficients in  $\sum_{\mu_i \leq \ell < \mu_j} \mathbf{C}z^\ell$ .

### Example

If  $A_0 := \begin{pmatrix} a & 0 \\ 0 & bz \end{pmatrix}, a, b \in \mathbf{C}^*$ , then the normal form of

$A_u := \begin{pmatrix} a & u \\ 0 & bz \end{pmatrix}, u \in K$  is  $\begin{pmatrix} a & \mathcal{B}_{q,1}u(a/b) \\ 0 & bz \end{pmatrix}$ , where:

$$\mathcal{B}_{q,1} \left( \sum f_n z^n \right) = \sum \frac{f_n}{q^{n(n-1)/2}} z^n.$$

# Irregular equations

## Formal isomorphism

Call  $\mathfrak{G} \subset \mathrm{GL}_n$  the subgroup of matrices:

$$\begin{pmatrix} I_{r_1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & F_{i,j} & \cdots \\ \cdots & 0 & \cdots & \cdots \\ 0 & \cdots & \cdots & I_{r_k} \end{pmatrix}$$

For all  $A$  in the formal class  $A_0$ , there is a unique  $\hat{F} \in \mathfrak{G}(\hat{K})$  such that  $\hat{F}[A_0] = A$ ; call it  $\hat{F}_A$ . Then:

$$A \sim A' \iff \hat{F}_{A'}(\hat{F}_A)^{-1} \in \mathfrak{G}(K).$$

We want to “sum” the divergent series  $\hat{F}_A$

### Example

$\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$  is an isomorphism from  $A_0$  to  $A_u$  if, and only if,

$bz\sigma_q f - af = u$ . This has a unique formal solution  $\hat{f}_u$ , and  $A_u \sim A_v \iff \hat{f}_u - \hat{f}_v \in K$ .

# Irregular equations

## $q$ -adapted Poincaré asymptotics

There is a  $q$ -analogue of Poincaré asymptotics with the following features:

### Asymptotics for ODE

1. Dynamics is given by the semi-group  $\Sigma := e^{[-\infty, 0]}$ .
2.  $\Sigma$ -invariant subsets of  $(\mathbb{C}^*, 0)$  are sectors.
3. The horizon  $(\mathbb{C}^*, 0)/\Sigma$  is the circle of directions  $S^1$ .
4. Sheaves are defined over  $S^1$ .

### Asymptotics for $q$ -differences

- ▶ Dynamics is given by the semi-group  $\Sigma := q^{-\mathbb{N}}$ .
- ▶  $\Sigma$ -invariant subsets of  $(\mathbb{C}^*, 0)$  are spiral-like.
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We write  $\mathcal{A}$  the sheaf of functions with an asymptotic expansion and:

$$\Lambda_I(M_0) := \mathfrak{B}(\mathcal{A}) \cap \text{Aut}(M_0),$$

the sheaf of automorphisms of  $M_0$  infinitely tangent to identity.

Actually, if  $\mathcal{A}_0$  denotes the subsheaf of flat functions:

$$\Lambda_I(M_0) \subset I_n + \text{GL}_n(\mathcal{A}_0),$$

whence the name.

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# Irregular equations

## Meromorphic summation

The polar divisor of a meromorphic isomorphism  $F : A_0 \rightarrow A$ , is  $q$ -invariant near 0, hence defined over  $\mathbf{C}^*/q^{-\mathbf{N}} = \mathbf{E}_q$ .

### Theorem

*There is an explicit finite subset  $\Sigma_{A_0} \subset \mathbf{E}_q$  such that, for all  $\bar{c} \in \mathbf{E}_q \setminus \Sigma_{A_0}$ , and for all  $A$ , there is a unique meromorphic isomorphism  $F : A_0 \rightarrow A_U$  such that:*

$$\forall 1 \leq i < j \leq k, \operatorname{div}_{\mathbf{E}_q}(F_{i,j}) \geq -(\mu_j - \mu_i)[-\bar{c}].$$

*We write  $S_{\bar{c}}\hat{F}_A$  this  $F$  and see it as a “resummation of  $\hat{F}_A$  in the (allowed) direction  $\bar{c} \in \mathbf{E}_q \setminus \Sigma_{A_0}$ ”. One has moreover:*

$$S_{\bar{c}}\hat{F}_A \sim \hat{F}_A.$$

# Irregular equations

Privileged cocycles of  $\Lambda_I(M_0)$

We note:

$$S_{c,d} \hat{F}_A := (S_{\bar{c}} \hat{F}_A)^{-1} (S_{\bar{d}} \hat{F}_A)$$

Properties:

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3.  $S_{c,d} \hat{F}_A - I_n$  is "flat".
4.  $\text{div}_{\mathbf{E}_q}((S_{c,d} \hat{F}_A)_{i,j}) \geq -(\mu_j - \mu_i)([-c] + [-d])$ .

Thus the  $S_{c,d} \hat{F}_A$  form a *privileged* cocycle of  $\Lambda_I(M_0)$  for the covering  $\mathcal{U}_{A_0}$  of  $\mathbf{E}_q$  made up of the Zariski open subsets  $V_{\bar{c}} := \mathbf{E}_q \setminus \{\bar{c}\}$ ,  $\bar{c} \in \mathbf{E}_q \setminus \Sigma_{A_0}$ .

# Irregular equations

Privileged cocycles of  $\Lambda_l(M_0)$

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# Irregular equations

## The $q$ -Malgrange-Sibuya theorems

An analogue of  
Stokes  
phenomenon for  
 $q$ -difference  
equations

Jacques Sauloy

Write  $Z_{pr}^1(\mathfrak{A}_{A_0}, \Lambda_I(M_0))$  the space of privileged cocycles.

### Theorem

*“Meromorphic summation” yields natural isomorphisms:*

$$\mathcal{F}(P_1, \dots, P_k) \simeq Z_{pr}^1(\mathfrak{A}_{A_0}, \Lambda_I(M_0)) \simeq H^1(\mathbf{E}_q, \Lambda_I(M_0)).$$

It is an easy (and pleasant) exercise to compute the dimension of  $Z_{pr}^1(\mathfrak{A}_{A_0}, \Lambda_I(M_0))$ .

Generalities

Slopes

Classification



# Irregular equations

Déviage  $q$ -Gevrey

“Abelian” case: two slopes  $\mu_1 < \mu_2$ , one “level”  $\delta := \mu_2 - \mu_1$ .

Then  $\Lambda_I(M_0)$  is an “elementary” vector bundle of slope  $\delta$  over  $\mathbf{E}_q$ :

$\Lambda_I(M_0) \simeq (\text{flat bundle of rank } r_1 r_2) \otimes (\text{line bundle of degree } \delta)$ .

General case: slopes  $\mu_1 < \dots < \mu_k$ , levels  $\mu_j - \mu_i, i < j$ .

The subsheaf  $\Lambda_I^t(M_0)$  made up of  $F$  s.t.  $F - I_n$  is  $t$ -flat has only diagonals  $\mu_j - \mu_i \geq t$ .

$\Lambda_I(M_0)$  is built from central extensions by elementary bundles  $\lambda_I^{(t)}(M_0)$ :

$$0 \rightarrow \lambda_I^{(t)}(M_0) \rightarrow \frac{\Lambda_I(M_0)}{\Lambda_I^{t+1}(M_0)} \rightarrow \frac{\Lambda_I(M_0)}{\Lambda_I^t(M_0)} \rightarrow 1.$$

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# Irregular equations

Two slopes, one level

$$\mathcal{F}(M_0) \simeq \text{Ext}(P_2, P_1) \simeq \text{Ext}(\underline{1}, P_2^\vee \otimes P_1) \simeq H^1(\mathbf{E}_q, \Lambda_I(M_0)).$$

## Example

$$\text{Let } A_0 := \begin{pmatrix} a & 0 \\ 0 & bz^\delta \end{pmatrix} \implies \Sigma_{A_0} = \{\bar{c} \in \mathbf{E}_q \mid c^\delta \in q^{\mathbb{Z}} a/b\}.$$

Components over  $V_{\bar{c}} \cap V_{\bar{d}}$  of cocycles of  $Z_{pr}^1(\mathcal{A}_{A_0}, \Lambda_I(M_0))$

are matrices  $S_{c,d} \hat{F}_A = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$ , where:

$$f(z) = \frac{g(z)}{\theta_q(z/c)^\delta \theta_q(z/d)^\delta},$$

$$g \in \mathcal{O}(\mathbf{C}^*) \text{ s.t. } \sigma_q g = (a/b)(z/cd)^\delta g.$$