Regular Sequences

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Outline

Everyone's favorite sequence

Thue–Morse sequence:

$$a(n) = \begin{cases} 0 & \text{if the binary representation of } n \text{ has an even number of 1s} \\ 1 & \text{if the binary representation of } n \text{ has an odd number of 1s.} \end{cases}$$

For $n \geq 0$, the Thue–Morse sequence is

$$0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\,0\,1\,0\,0\,1\,0\,1\,1\,0\,0\,1\,1\,0\,1\,0\,0\,1\,\cdots.$$

Rediscovered several times as an infinite cube-free word on $\{0, 1\}$.

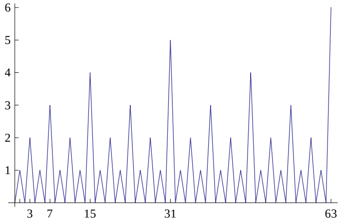
$$a(2n+0) = a(n)$$
$$a(2n+1) = 1 - a(n)$$

My favorite sequence

Let $\nu_k(n)$ be the exponent of the largest power of k dividing n.

For $n \geq 0$, the "ruler sequence" $\nu_2(n+1)$ is

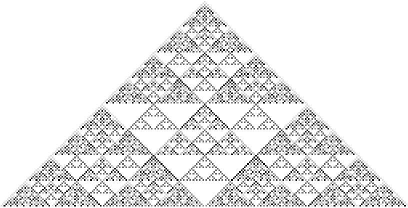
$$0\,1\,0\,2\,0\,1\,0\,3\,0\,1\,0\,2\,0\,1\,0\,4\,0\,1\,0\,2\,0\,1\,0\,3\,0\,1\,0\,2\,0\,1\,0\,5 \cdots .$$

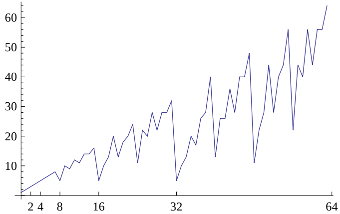


$$\nu_2(2n+0) = 1 + \nu_2(n)$$
$$\nu_2(2n+1) = 0$$

Counting nonzero binomial coefficients modulo 8

Let $a(n) = |\{0 \le m \le n : \binom{n}{m} \not\equiv 0 \mod 8\}|$.





1 2 3 4 5 6 7 8 5 10 9 12 11 14 14 16 5 10 13 20 13 18 20 24 $\cdots$

$$
\begin{aligned}
a(2n+1) &= 2a(n) \\
a(4n+0) &= a(2n) \\
a(8n+2) &= -2a(n) + 2a(2n) + a(4n+2) \\
a(8n+6) &= 2a(4n+2)
\end{aligned}
$$

Definition

Convention:
We index sequences starting at $n = 0$.

Definition (Allouche & Shallit 1992)

Let $k \geq 2$ be an integer. An integer sequence $a(n)$ is *k-regular* if the $\mathbb{Z}$-module generated by the set of subsequences

$$\{a(k^e n + i) : e \geq 0, 0 \leq i \leq k^e - 1\}$$

is finitely generated.

We can take the generators to be elements of this set.
Every $a(k^e n + i)$ is a linear combination of the generators.

In particular, $a(k^e(kn + j) + i)$ is a linear combination of the generators, which gives a finite set of recurrences that determine $a(n)$.

Homogenization

For the Thue–Morse sequence:

$$a(2n+0) = a(n)$$
$$a(2n+1) = 1 - a(n)$$

But we can homogenize:

$$a(2n) = a(n)$$
$$a(2n+1) = a(2n+1)$$
$$a(4n+1) = a(2n+1)$$
$$a(4n+3) = a(n)$$

So $a(n)$ and $a(2n+1)$ generate the $\mathbb{Z}$-module, and we have written $a(2n+0), a(2n+1), a(2(2n+0)+1), a(2(2n+1)+1)$ as linear combinations of the generators.

Basic properties

Regular sequences inherit self-similarity from base-k representations of integers.

The nth term $a(n)$ can be computed quickly — using $O(\log n)$ additions and multiplications.

The set of k-regular sequences is closed under…

- termwise addition
- termwise multiplication
- multiplication as power series
- shifting ($b(n) = a(n+1)$)
- modifying finitely many terms

Outline

More examples

Regular sequences are everywhere...

- The length $a(n)$ of the base-k representation of $n+1$ is a k-regular sequence:

$$a(kn+i) = 1 + a(n).$$

- The number of comparisons $a(n)$ required to sort a list of length n using merge sort is

$$0\ 0\ 1\ 3\ 5\ 8\ 11\ 14\ 17\ 21\ 25\ 29\ 33\ 37\ 41\ 45\ \cdots.$$

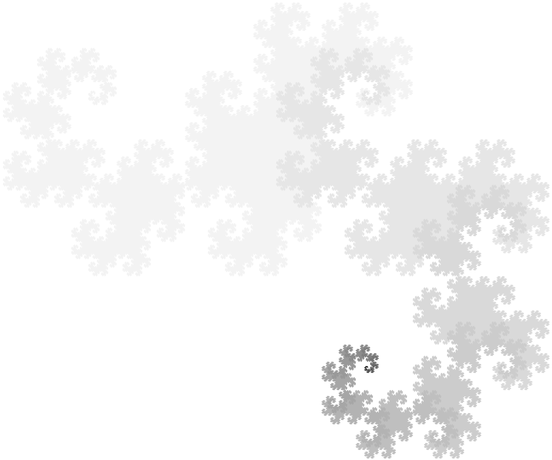
  This sequence satisfies

$$a(n) = a(\lceil \tfrac{n}{2} \rceil) + a(\lfloor \tfrac{n}{2} \rfloor) + n - 1$$

  and is 2-regular.

Dragon curve

The coordinates $(x(n), y(n))$ of paperfolding curves are 2-regular.



p-adic valuations of integer sequences

- $\nu_k(n+1)$ is k-regular:

$$\begin{aligned} a(kn+k-1) &= 1 + a(n) \\ a(kn+i) &= 0 \qquad \text{if } i \neq k-1. \end{aligned}$$

- Bell 2007:
  If $f(x)$ is a polynomial, $\nu_p(f(n))$ is p-regular if and only if $f(x)$ factors as
  (product of linear polynomials over $\mathbb{Q}$) $\cdot$ (polynomial with no roots in $\mathbb{Z}_p$).
- $\nu_p(n!)$ is p-regular.
- Closure properties imply that

$$\nu_p(C_n) = \nu_p((2n)!) - 2\nu_p(n!) - \nu_p(n+1)$$

  is p-regular.

p-adic valuations of integer sequences

Medina–Rowland 2009:
$\nu_p(F_n)$ is p-regular.

The Motzkin numbers M_n satisfy

$$(n+2)M_n - (2n+1)M_{n-1} - 3(n-1)M_{n-2} = 0.$$

Conjecture

If $p = 2$ or $p = 5$, then $\nu_p(M_n)$ is p-regular.

Open question

Given a polynomial-recursive sequence $f(n)$, for which primes is $\nu_p(f(n))$ p-regular?

"Number theoretic combinatorics"

- The sequence of integers expressible as a sum of distinct powers of 3 is 2-regular:

$$0\ 1\ 3\ 4\ 9\ 10\ 12\ 13\ 27\ 28\ 30\ 31\ 36\ 37\ 39\ 40\ \cdots$$

$$\begin{aligned} a(2n) &= 3a(n) \\ a(4n+1) &= 6a(n) + a(2n+1) \\ a(4n+3) &= -3a(n) + 4a(2n+1) \end{aligned}$$

- The sequence of integers whose binary representations contain an even number of 1s is 2-regular:

$$0\ 3\ 5\ 6\ 9\ 10\ 12\ 15\ 17\ 18\ 20\ 23\ 24\ 27\ 29\ 30\ \cdots$$

- Let $|n|_w$ be the number of occurrences of w in the base-k representation of n. For every word w, $|n|_w$ is k-regular.

Nonzero binomial coefficients

Let $a_{p^\alpha}(n) = |\{0 \le m \le n : \binom{n}{m} \not\equiv 0 \mod p^\alpha\}|$.

- Glaisher 1899:
$$a_2(n) = 2^{|n|_1}.$$
- Fine 1947:
$$a_p(n) = \prod_{i=0}^{l} (n_i + 1),$$
where $n = n_l \cdots n_1 n_0$ in base p.

  For example, $a_5(n) = 2^{|n|_1} 3^{|n|_2} 4^{|n|_3} 5^{|n|_4}$.

It follows that $a_p(n)$ is p-regular.

Nonzero binomial coefficients

Rowland 2011:
Algorithm for obtaining a symbolic expression in n for $a_{p^\alpha}(n)$.
It follows that $a_{p^\alpha}(n)$ is p-regular for each $\alpha \geq 0$.

For example:

$$a_{p^2}(n) = \left(\prod_{i=0}^{l} (n_i + 1)\right) \cdot \left(1 + \sum_{i=0}^{l-1} \frac{p - (n_i + 1)}{n_i + 1} \cdot \frac{n_{i+1}}{n_{i+1} + 1}\right).$$

Expressions for $p = 2$ and $p = 3$:

$$a_4(n) = 2^{|n|_1} \left(1 + \frac{1}{2}|n|_{10}\right)$$

$$a_9(n) = 2^{|n|_1} 3^{|n|_2} \left(1 + |n|_{10} + \frac{1}{4}|n|_{11} + \frac{4}{3}|n|_{20} + \frac{1}{3}|n|_{21}\right)$$

Nonzero binomial coefficients

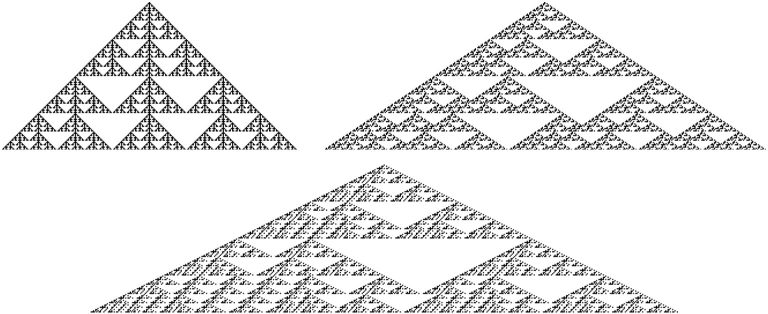
Higher powers of 2:

$$a_8(n) = 2^{|n|_1} \left(1 + \frac{1}{8}|n|_{10}^2 + \frac{3}{8}|n|_{10} + |n|_{100} + \frac{1}{4}|n|_{110}\right)$$

$$\begin{aligned}\frac{a_{16}(n)}{2^{|n|_1}} = 1 + \frac{5}{12}|n|_{10} + \frac{1}{2}|n|_{100} + \frac{1}{8}|n|_{110} \\ + 2|n|_{1000} + \frac{1}{2}|n|_{1010} + \frac{1}{2}|n|_{1100} + \frac{1}{8}|n|_{1110} + \frac{1}{16}|n|_{10}^2 \\ + \frac{1}{2}|n|_{10}|n|_{100} + \frac{1}{8}|n|_{10}|n|_{110} + \frac{1}{48}|n|_{10}^3\end{aligned}$$

Powers of polynomials

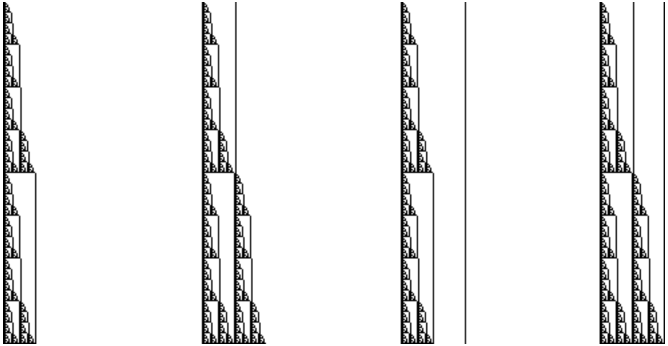
If $f(x) \in \mathbb{F}_{p^\alpha}[x]$, how many nonzero terms are there in $f(x)^n$?
Such a sequence has an interpretation as counting cells in a cellular automaton. Here is $(x^d + x + 1)^n$ over $\mathbb{F}_2$ for $d = 2, 3, 4$:



Amdeberhan–Stanley ~2008:
Let $f(x_1, \ldots, x_m) \in \mathbb{F}_{p^\alpha}[x_1, \ldots, x_m]$. The number $a(n)$ of nonzero terms in the expanded form of $f(x_1, \ldots, x_m)^n$ is p-regular.

Another kind of self-similarity

Here is a cellular automaton that grows like $\sqrt{n}$:



The length of row n is 2-regular.

The number of black cells on row n is 2-regular.

Lexicographically extremal words avoiding a pattern

What is the lexicographically least square-free word on $\mathbb{Z}_{\geq 0}$?

$$0\,1\,0\,2\,0\,1\,0\,3\,0\,1\,0\,2\,0\,1\,0\,4\,0\,1\,0\,2\,0\,1\,0\,3\,0\,1\,0\,2\,0\,1\,0\,5\cdots$$

The nth term is $\nu_2(n+1)$.

The lexicographically least k-power-free word is given by $\nu_k(n+1)$.

$k = 3$:

$$0\,0\,1\,0\,0\,1\,0\,0\,2\,0\,0\,1\,0\,0\,1\,0\,0\,2\,0\,0\,1\,0\,0\,1\,0\,0\,3\,0\,0\,1\,0\,0\cdots$$

$k = 4$:

$$0\,0\,0\,1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\,0\,2\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\,0\,2\cdots$$

Lexicographically extremal words avoiding a pattern

If $w = w_1 w_2 \cdots w_l$ is a length-l word and $r \in \mathbb{Q}_{\geq 0}$ such that $r \cdot l \in \mathbb{Z}$, let

$$w^r = w^{\lfloor r \rfloor} w_1 w_2 \cdots w_{l \cdot (r - \lfloor r \rfloor)}$$

be the word consisting of repeated copies of w truncated at rl letters.

For example...

$$(\text{deci})^{3/2} = \text{decide}$$

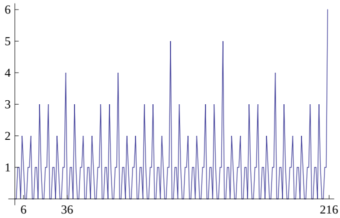
$$(\text{raisonne})^{3/2} = \text{raisonnerais}$$

$$(\text{schuli})^{3/2} = \text{schulisch}$$

Lexicographically extremal words avoiding a pattern

What is the lexicographically least word on $\mathbb{Z}_{\geq 0}$ avoiding $3/2$-powers?

$$0\,0\,1\,1\,0\,2\,1\,0\,0\,1\,1\,2\,0\,0\,1\,1\,0\,3\,1\,0\,0\,1\,1\,3\,0\,0\,1\,1\,0\,2\,1\,0\,0\,1\,1\,4\,\cdots$$



Rowland–Shallit 2011:
This sequence is 6-regular.

Open question

When are such sequences k-regular, and for what value of k?

Outline

Constant-recursive sequences

The companion matrix of a constant-recursive sequence $a(n)$ satisfies

$$M \cdot \begin{bmatrix} a(n) \\ a(n+1) \\ \vdots \\ a(n+r-1) \end{bmatrix} = \begin{bmatrix} a(n+1) \\ a(n+2) \\ \vdots \\ a(n+r) \end{bmatrix}.$$

For example,

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n + F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_{n+2} \end{bmatrix}.$$

So

$$F_n = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In general $a(n) = \lambda M^n \kappa$.

Matrix formulation

Take r generators $a_1(n), \ldots, a_r(n)$ of a k-regular sequence.
Each $a_j(kn + i)$ is a linear combination of the r generators.

Encode the coefficients in $r \times r$ matrices $M_0, M_1, \ldots, M_{k-1}$.
Then if $n = n_l \cdots n_1 n_0$ in base k, then

$$a(n) = \lambda M_{n_l} \cdots M_{n_1} M_{n_0} \kappa.$$

Again consider the Thue–Morse sequence; generators $a(n), a(2n + 1)$.

$$\begin{aligned} a(2n) &= 1 \cdot a(n) + 0 \cdot a(2n+1) \\ a(2n+1) &= 0 \cdot a(n) + 1 \cdot a(2n+1) \\ a(2(2n+0)+1) &= 0 \cdot a(n) + 1 \cdot a(2n+1) \\ a(2(2n+1)+1) &= 1 \cdot a(n) + 0 \cdot a(2n+1) \end{aligned}$$

Then

$$M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Matrix formulation

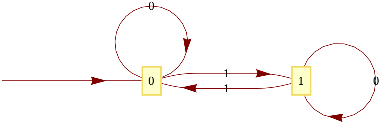
Corollaries...

- In a sense, constant-recursive sequences are "1-regular".
- A k-regular sequence has constant-recursive subsequences. For example, $a(k^n) = \lambda M_1 M_0^n \kappa$.
- If $a(n)$ is k-regular, then $a(n) = O(n^d)$ for some d.

Automatic sequences

A sequence $a(n)$ is *k-automatic* if there is a finite automaton whose output is $a(n)$ when fed the base-k digits of n.

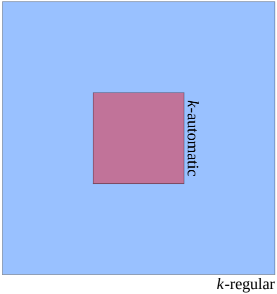
The Thue–Morse sequence is 2-automatic:



Allouche–Shallit 1992:
A k-regular sequence is finite-valued if and only if it is k-automatic.

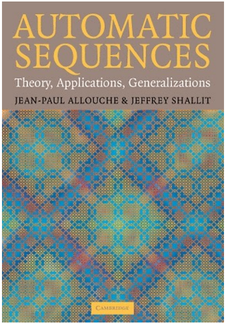
Hierarchy of integer sequences

Fix $k \geq 2$.



Automatic sequences

Automatic sequences have been very well studied.

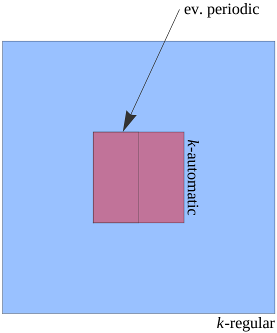


Charlier–Rampersad–Shallit 2011:
Many operations on k-automatic sequences produce k-regular sequences.

Büchi 1960:
If $a(n)$ is eventually periodic, then $a(n)$ is k-automatic for every $k \geq 2$.

Hierarchy of integer sequences

We add eventually periodic sequences:



Polynomials

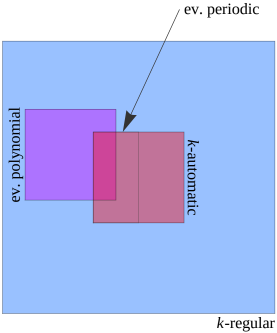
The sequence $a(n) = n$ is k-regular for every $k \geq 2$:

$$a(kn + i) = k(1 - i)a(n) + i\,a(kn + 1)$$
$$a(k^2n + i) = k(k - i)a(n) + i\,a(kn + 1)$$

It follows that every polynomial sequence is k-regular
(as every polynomial sequence is constant-recursive).

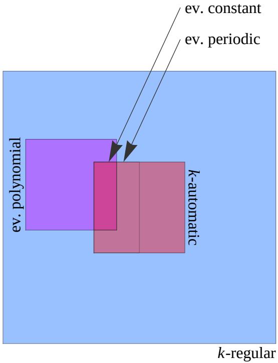
Hierarchy of integer sequences

Every (eventually) polynomial sequence is k-regular.



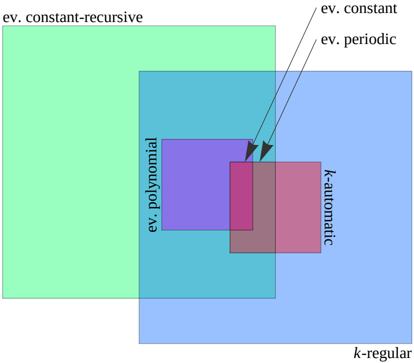
Hierarchy of integer sequences

If $a(n)$ is eventually polynomial and k-automatic, then $a(n)$ is eventually constant.



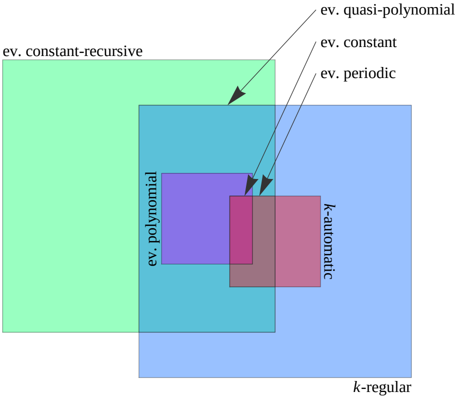
Hierarchy of integer sequences

Every polynomial sequence is constant-recursive.
(And not every k-automatic sequence is constant-recursive.)



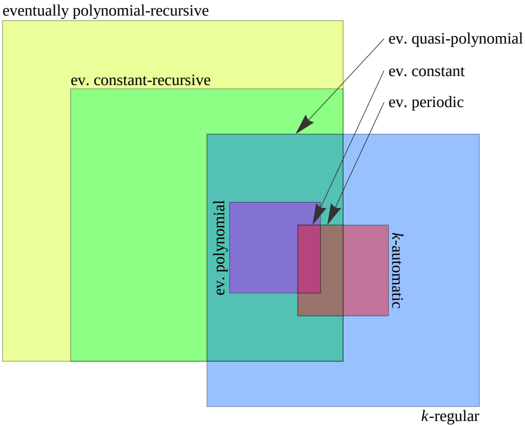
Hierarchy of integer sequences

Allouche–Shallit 1992:
If $a(n)$ is constant-recursive and k-regular, then $a(n)$ is eventually quasi-polynomial.



Hierarchy of integer sequences

And to entice us. . .



Sequences that are not regular

- By Bell's theorem, $\nu_2(n^2 + 7)$ is not 2-regular.

  0 3 0 4 0 5 0 3 0 3 0 7 0 4 0 3 0 3 0 4 0 6 0 3 0 3 0 5 0 4 0 3 $\cdots$

- Bell ~2005, Moshe 2008, Rowland 2010:
  $\lfloor \alpha + \log_k(n+1) \rfloor$ is k-regular if and only if k^α is rational.
  For example, $\lfloor \frac{1}{2} + \log_2(n+1) \rfloor$ is not 2-regular.

  0 1 2 2 2 3 3 3 3 3 3 4 4 4 4 4 4 4 4 4 4 4 5 5 5 5 5 5 5 5 5 5 $\cdots$

Is there a natural (larger) class that these sequences belong to?

Generalizations

Two generalizations of k-regular sequences:

- Allow polynomial coefficients in n (analogous to polynomial-recursive sequences).
- Becker 1994, Dumas 1993, Randé 1992:
  If $a(n)$ is k-regular, then $f(x) = \sum_{n=0}^{\infty} a(n) x^n$ satisfies a Mahler functional equation
  $$\sum_{i=0}^{m} p_i(x) f(x^{k^i}) = 0.$$

How natural are these generalizations?

It remains to be seen…