Mean asymptotic behaviour of radix-rational sequences and dilation equations

Philippe Dumas,
ALGORITHMS, INRIA PARIS-ROCQUENCOURT

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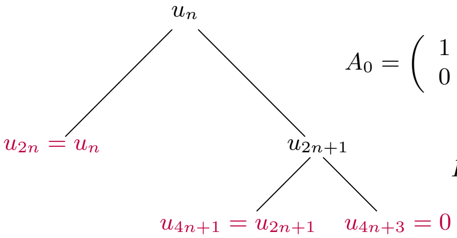
What is a radix-rational sequence (aka a k-regular sequence)?

- A very simple example

$$u_n = \begin{cases} 1 & \text{if } n \text{ is a power of } 2 \\ 0 & \text{otherwise} \end{cases}$$



$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$L = \begin{pmatrix} 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- Formal definition A (complex) sequence is rational with respect to radix B if there is a finite dimensional vector space which contains the sequence and is left stable by the B-section operators. (Allouche and Shallit, 1992)
- Linear representation A sequence is B-rational if and only if it admits a linear representation A_0, A_1, $\ldots$, A_{B-1}, L, C.

- Domains
  - binary coding of integers (sum of digits, Thue-Morse sequence, Rudin-Shapiro sequence…)
  - theory of numbers (Pascal triangle reduced modulo a power of 2, sum of three squares)
  - divide-and-conquer algorithms (binary powering, Euclidean matching…)
- A more natural example Cost of mergesort in the worst case

$$u_n = u_{\lceil n/2 \rceil} + u_{\lfloor n/2 \rfloor} + n - 1, \quad u_0 = 0, \quad u_1 = 0$$

$$B = 2 \quad A_0 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 3 \end{pmatrix}$$

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{tr}}$$

$$13 = (1101)_2 \quad u_{13} = LA_1A_1A_0A_1C$$

- Classical case $B = 1$, $u_n = LA_0^n C$

Asymptotic behaviour of radix-rational sequences

- Theorem Each B-rational sequence admits an asymptotic expansion of the form

$$u_n \underset{n\to+\infty}{=} \sum_{\alpha>\alpha_*,\ell\geq 0} n^\alpha \log_B^\ell(n) \sum_\omega \omega^{\lfloor \log_B n\rfloor} \Psi_{\alpha,\ell,\omega}(\log_B n) + O(n^{\alpha_*})$$

  ω modulus 1 complex number, Ψ 1-periodic function

- Example Worst mergesort: $u_n = n\log_2 n + n\Psi(\log_2 n) + 1$, with $\Psi(t) = 1 - \{t\} - 2^{1-\{t\}}$ (Flajolet and Golin, 1994)
- Average or not? Study of $\displaystyle\sum_{0\leq n\leq N} u_n$
- Tools
  - rational formal power series
  - dilation equation
  - joint spectral radius
  - Jordan reduction
  - numeration system

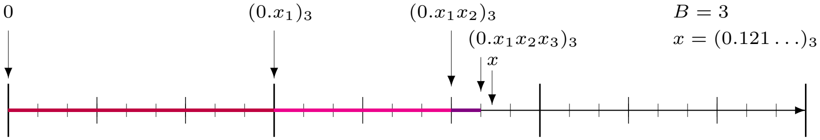
Rational formal power series

- Radix-rational sequence and rational formal power series
  - Every radix-rational sequence hides a rational formal power series.
    alphabet $\mathcal{B} = \{0, 1, \ldots, B-1\}$
    formal power series $S = \displaystyle\sum_{w \in \mathcal{B}^*} (S, w) w$
    here $(S, w) = L A_{w_1} A_{w_2} \cdots A_{w_K} C = L A_w C$ if $w = w_1 w_2 \cdots w_K$
  - Every rational formal power series defines a radix-rational sequence.
    $n = (w)_B$, $u_n = (S, w)$
  - The rational formal power series is the essential object.

- Running sum

$$\mathbf{S}_K(x) = \sum_{\substack{|w|=K \\ (0.w)_B \le x}} A_w C \qquad\qquad Q = A_0 + A_1 + \cdots + A_{B-1}$$

$$\mathbf{S}_K(x) = \sum_{r_1 < x_1} A_{r_1} Q^{K-1} C + \sum_{r_2 < x_2} A_{x_1} A_{r_2} Q^{K-2} C + \sum_{r_3 < x_3} A_{x_1} A_{x_2} A_{r_3} Q^{K-3} C + \cdots + \sum_{r_K \le x_K} A_{x_1} A_{x_2} \cdots A_{r_K} C$$



Lemma

With $Q = A_0 + A_1 + \cdots + A_{B-1}$, *the sequence of running sums* $(\mathbf{S}_K)$ *satisfies the recursion*

$$\mathbf{S}_{K+1}(x) = \sum_{r_1 < x_1} A_{r_1} Q^K C + A_{x_1} \mathbf{S}_K(Bx - x_1),$$

where x_1 *is the first digit in the radix-*B *expansion of* x *in* $[0, 1)$, *with* $\mathbf{S}_0(x) = C$.

Dilation equation

- Basic case

  Hypothesis: $Q^K C \underset{K\to+\infty}{=} R(K)\left(V + O\left(\frac{1}{K}\right)\right)$
  for some nonzero vector V with $R(K+1)/R(K) = \rho\omega(1+O(1/K))$ and $\rho > 0$, $|\omega| = 1$.

$$\mathbf{F}_K(x) = \frac{1}{R(K)}\mathbf{S}_K(x), \qquad \mathbf{F}_{K+1} = \mathcal{L}_K \mathbf{F}_K$$

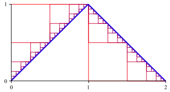
$$\mathcal{L}_K \Phi(x) = \frac{1}{R(K+1)} \sum_{r_1 < x_1} A_{r_1} Q^K C + \frac{R(K)}{R(K+1)} A_{x_1} \Phi(Bx - x_1)$$

  Basic dilation equation:
  - $\Phi(0) = 0$, $\Phi(1) = V$,
  - for every digit r of the radix-B system and for x in $[r/B, (r+1)/B)$,

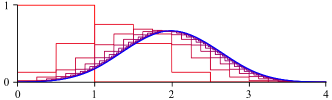
$$\Phi(x) = \frac{1}{\rho\omega} \sum_{r_1 < r} A_{r_1} V + \frac{1}{\rho\omega} A_r \Phi(Bx - r).$$

- Wavelets
  - Scaling function φ (or father wavelet), data (c_k) with hypotheses
    Daubechies, 1988 : There exists a unique function $\varphi \in L^2(\mathbb{R})$ such that
    1. $\varphi(x) = \sum_{k=0}^{K-1} c_k \varphi(2x - k)$
    2. $\int_{\mathbb{R}} \varphi(x)\, dx = 1$
    3. $\operatorname{supp} \varphi \subset [0, K-1]$
  - Mother wavelet $\psi(x) = \sum_k (-1)^k c_{g-1-k} \varphi(2x - k)$

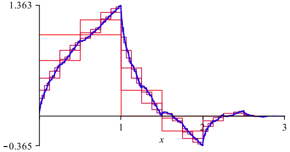
    $$\text{Wavelets } \varphi_k(x) = \varphi(x-k), \qquad \psi_{j,k}(x) = 2^{-j/2}\psi(2^{-j}x - k)$$
  - Expansion $f = \sum_k \langle f, \varphi_k \rangle \varphi_k + \sum_j \sum_k \langle f, \psi_{j,k} \rangle \psi_{j,k}$ for $f \in L^2(\mathbb{R})$
  - Example: iteration from the box function (contracting operator)



hat $\quad c_0 = 1/2,\ c_1 = 1,\ c_2 = 1/2$



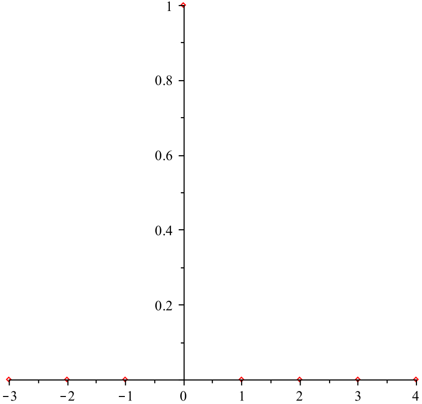
cubic B-spline $\quad c_0 = 1/8,\ c_1 = 4/8,\ c_2 = 6/8,\ c_3 = 4/8,\ c_4 = 1/8$



Daubechies

$$c_0 = (1+\alpha)/4,\quad c_1 = (3+\alpha)/4,\quad c_2 = (3-\alpha)/4,\quad c_3 = (1-\alpha)/4,\quad \alpha = \sqrt{3}$$

- Refinement schemes (Deslauriers and Dubuc, 1986)
  - Interpolation scheme (gliding Lagrange interpolation)
    data: $(v_k)_{k\in\mathbb{Z}}$ and $L>0$,
    output: f function such that $f(k)=v_k$
    if f defined on $\dfrac{1}{2^j}\mathbb{Z}$ and $x_{j,k}=\dfrac{k}{2^j}+\dfrac{1}{2^{j+1}}$ then $f(x_{j,k})=\pi_{j,k}(x_{j,k})$ where $\pi_{j,k}$ is the Lagrange interpolation polynomial at $p/2^j$ with $k-L\le p\le k+L+1$
  - Correction If (v_k) bounded, f extends to $\mathbb{R}$ as a continuous function
  - Scaling function φ
    $v_0=1$, $v_k=0$ for $k\neq 0$
    $$f(x)=\sum_{k\in\mathbb{Z}} v_k\varphi(x-k)$$
    $$\varphi(x)=\sum_k c_k\varphi(2x-k)$$
  - Example with $L=2$ (cascade algorithm)



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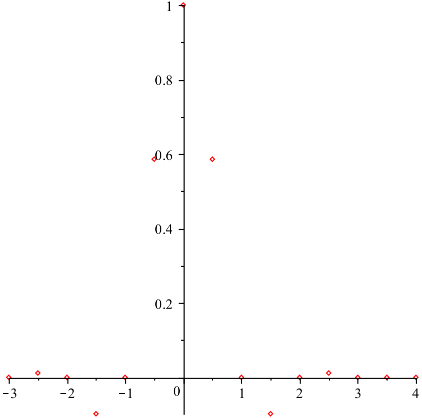

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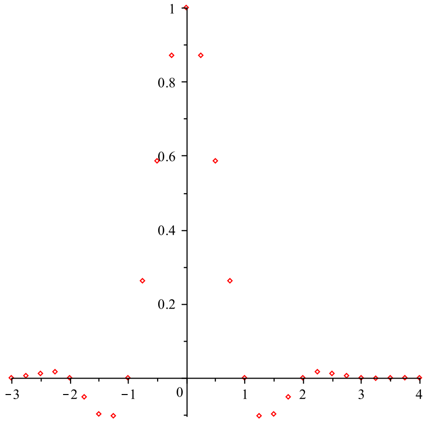

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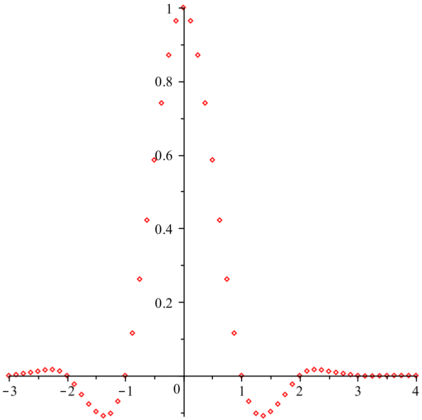
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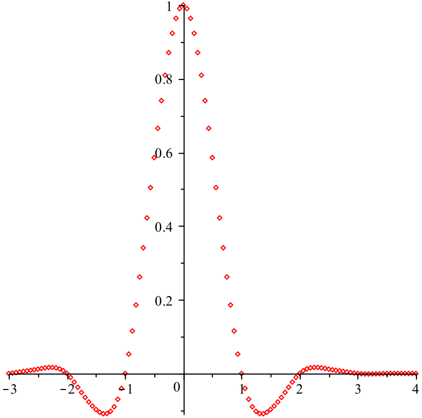

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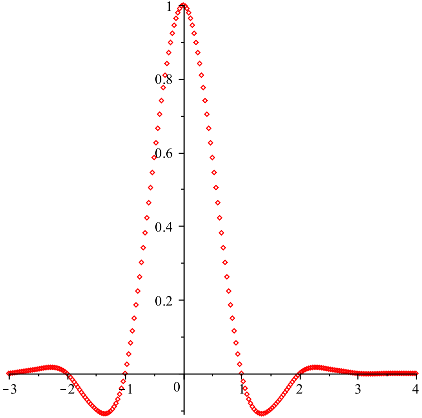

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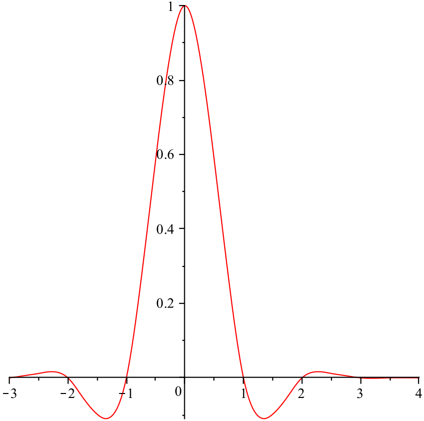
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- Basic result

Theorem

Let L, $(A_r)_{0\leq r<B}$, C be a linear representation of dimension d for the radix B. It is assumed that

- $Q^K C \underset{K\to+\infty}{=} R(K)\left(V+O\left(\frac{1}{K}\right)\right)$ *for some nonzero vector V with* $R(K+1)/R(K)=\rho\omega(1+O(1/K))$ *and* $\rho>0$, $|\omega|=1$
- *there exists and induced norm $\|\ \|$ and a constant λ, with $0<\lambda<\rho$ such that all matrices A_r, $0\leq r<B$, satisfy $\|A_r\|\leq\lambda$.*

Then

- *the basic dilation equation has a unique solution $\mathbf{F}$, which is continuous from $[0,1]$ into $\mathbb{C}^d$,*
- *the sequence $(\mathbf{F}_K)$ converges uniformly towards $\mathbf{F}$, with speed essentially $O((\lambda/\rho)^K)$.*

Concretely $\mathbf{S}_K(x) \underset{K\to+\infty}{=} R(K)\mathbf{F}(x)+O(\lambda^K)$

- Contribution of dilation equations
  - Contracting operator
  - Cascade algorithm
  - Regularity $\mathbf{F}$ is Hölder with exponent $\log_B(\rho/\lambda)$
  - Form of the dilation equation $(B = 2)$
    - ★ Piecewise equation, non homogeneous

$$\mathbf{F}(x) = \begin{cases} T_0\mathbf{F}(2x) & \text{if } 0 \leq x \leq 1/2 \\ T_0 V + T_1\mathbf{F}(2x-1) & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

    - ★ Global equation, homogeneous

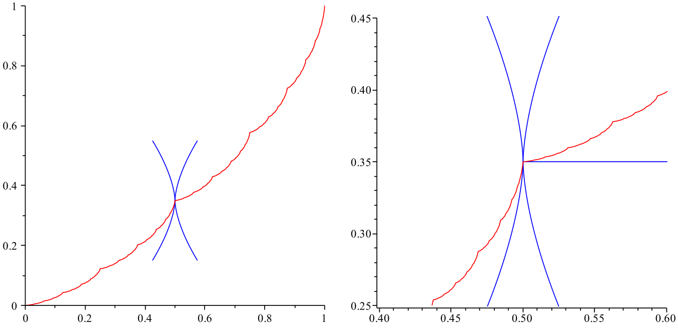
$$\mathbf{F}(x) = T_0\mathbf{F}(2x) + T_1\mathbf{F}(2x-1) \quad \text{for } x \text{ real}$$

with $\mathbf{F}$ constant on the left of 0 and on the right of 1

  - Example Billingsley's distribution functions (Billingsley, 1995)

$$X = \sum_{n\geq 0} \frac{X_n}{2^n},\ X_n = \text{Bernoulli}(p),\ 0 < p < 1$$

$L = \begin{pmatrix} 1 \end{pmatrix}$, $A_0 = \begin{pmatrix} 1-p \end{pmatrix}$, $A_1 = \begin{pmatrix} p \end{pmatrix}$, $C = \begin{pmatrix} 1 \end{pmatrix}$
$Q = \begin{pmatrix} 1 \end{pmatrix}$, $\rho = 1$, $\omega = 1$ $V = \begin{pmatrix} 1 \end{pmatrix}$, $R(K) = 1$,
F distribution function, $F(x) = (1-p)F(2x) + pF(2x-1)$, $F(0) = 0$, $F(1) = 1$



$\lambda = \max(p, 1-p)$
Hölder exponent $\alpha = \log_2(1/\max(p, 1-p)) \simeq 0.62$ (here $p = 13/20$)
if $1/2 < p < 1$

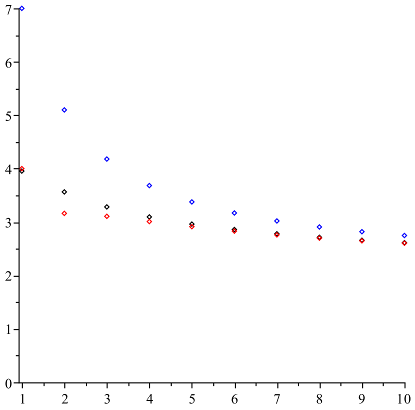
essentially best Hölder exponent on the right $\alpha_+ = \log_2(1/(1-p)) \simeq 1.51 > 1$

essentially best Hölder exponent on the left $\alpha_- = \alpha$

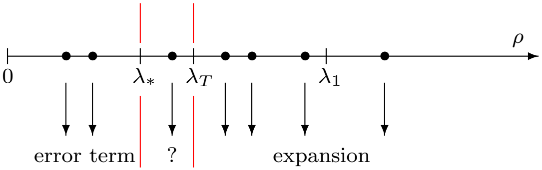
Joint spectral radius

- Controlling products A_w Rota and Strang, 1960: $\lambda_T = \max_{|w|=T} \|A_w\|^{1/T}$
  joint spectral radius $\lambda_* = \lim_{T\to+\infty} \lambda_T$

example with worstmergesort
$1 = \diamond$
$\infty = \diamond$
$2 = \diamond$



- Changing the radix
  - From B to B^T If a sequence is B-rational, it is B^T-rational forall $T \in \mathbb{N}_{>0}$.
    Linear representation L, A_r, C with $0 \leq r < B$ for B
    becomes L, A_w, C with $|w| = T$ for B^T.
  - Eigenvalues $Q = \sum_{0 \leq r < B} A_r$ becomes $Q^{(T)} = \sum_{|w|=T} A_w = Q^T$
    ρ becomes ρ^T (and $\mathbf{S}_K$ becomes $\mathbf{S}_{KT}$)
  - Dichotomy (desired)



- Jordan reduction
  - Idea
    Jordan reduction of matrix Q and processing of each generalized eigenspace
    Vector valued functions become matrix valued functions, but same arguments
  - Qualitative result

Theorem

Let L, $(A_r)_{0\le r<B}$, C be a linear representation of a formal power series S. The sequence of running sums

$$S_K(x) = L\mathbf{S}_K(x) = \sum_{\substack{|w|=K\\(0.w)_B\le x}} LA_wC$$

admits an asymptotic expansion with error term $O(\lambda^K)$ for every $\lambda > \lambda_*$, where λ_* is the joint spectral radius of the family $(A_r)_{0\le r<B}$. The used asymptotic scale is the family of sequences $\rho^K\binom{K}{\ell}$, $\rho > 0$, $\ell \in \mathbb{N}_{\ge 0}$. The coefficients are related to solutions of dilation equations. The error term is uniform with respect to $x \in [0,1]$.

- Numeration system

  We return to $\sum_{n=0}^{N} u_n$.

  - Idea

    $N = B^{K+t}$, $K = \lfloor \log_B N \rfloor$, $t = \{\log_B N\}$

    sum up to N =

    [sum of u_n up to $B^K - 1$] plus [sum of u_n from B^k to N]=

    [(sum of (S, w) for $|w| \leq K$)
    minus (sum for those words beginning with 0)]
    plus
    [(running sum of (S, w) up to N for words of length $K + 1$)
    minus (sum for those words beginning with 0)]

- Technique

$$\sum_{n \le B^{K+t}} u(n) = \sum_{0 \le k \le K} \left(\sum_{|w|=k} L A_w C - \sum_{|w'|=k-1} L A_0 A_{w'} C \right) + \left(\sum_{\substack{|w|=K+1 \\ (w)_B \le B^{K+1} B^{t-1}}} L A_w C - \sum_{|w'|=K} L A_0 A_{w'} C \right)$$

that is

$$\sum_{n \le B^{K+t}} u(n) = L(\mathrm{I}_d - A_0) \sum_{0 \le k \le K} Q^k C + \sum_{\substack{|w|=K+1 \\ (w)_B \le B^{K+1} B^{t-1}}} L A_w C$$

or

$$\sum_{n \le B^{K+t}} u(n) = L(\mathrm{I}_d - A_0) \sum_{0 \le k \le K} Q^k C + L\mathbf{S}_{K+1}(B^{t-1})$$

and we are at home.

Result and comments

- Qualitative result

Theorem

Let L, $(A_r)_{0\le r<B}$, C be a linear representation of a radix rational sequence (u_n). The running sum $\sum_{n=0}^{N} u_n$ admits an asymptotic expansion with error term $O(N^{\log_B\lambda})$ for every $\lambda>\lambda_*$, where λ_* is the joint spectral radius of the family $(A_r)_{0\le r<B}$. The used asymptotic scale is the family of sequences $N^{\alpha}\binom{\lfloor\log_B N\rfloor}{\ell}$, $\alpha\in\mathbb{R}$, $\ell\in\mathbb{N}_{\ge 0}$. The coefficients write $\omega^{\lfloor\log_B N\rfloor}\Phi(\log_B N)$ where ω is modulus 1 complex numer and $\Phi(t)$ is 1-periodic and related to some solution of a dilation equation by the change of variable $x=B^{\{t\}-1}$

$$\rho^K\omega^K\binom{K}{\ell}F(x)\longrightarrow N^{\log_B\rho}\binom{\lfloor\log_B N\rfloor}{\ell}\times\Phi(\log_B N)$$

$$\Phi(t)=\omega^{\lfloor t\rfloor}\rho^{1-\{t\}}F(B^{\{t\}-1})$$

- Example Discrepancy of the van der Corput sequence (Béjian and Faure, 1977)
  Van der Corput sequence: $n = (n_{\ell-1} \dots n_1 n_0)_2 \qquad u_n = (0.n_0 n_1 \dots n_{\ell-1})_2$
  Discrepancy:

  $$D(n) = \sup_{0\le\alpha<\beta\le1} \left| \frac{\nu(n,\alpha,\beta)}{n} - (\beta-\alpha) \right|,$$

  Béjian and Faure sequence: $E(n) = nD(n)$

  $$E(1) = 1, \qquad E(2n) = E(n), \qquad E(2n+1) = \frac{1}{2}(E(n) + E(n+1) + 1)$$

  basis $(E(n), E(n+1), 1)$, linear representation

  $$L = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/2 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1/2 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

  $$\lambda_* = 1 \qquad Q = \begin{pmatrix} 3/2 & 1/2 & 0 \\ 1/2 & 3/2 & 0 \\ 1/2 & 1/2 & 2 \end{pmatrix}$$

Jordan reduction, with basis (V_1, V_2^0, V_2^1)

$V_1 = \begin{pmatrix} 1/2 & -1/2 & 0 \end{pmatrix}^{\mathrm{tr}}, \qquad V_2^0 = \begin{pmatrix} 0 & 0 & 1/2 \end{pmatrix}^{\mathrm{tr}}, \qquad V_2^1 = \begin{pmatrix} 1/2 & 1/2 & 0 \end{pmatrix}^{\mathrm{tr}},$

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \qquad C = V_1 + V_2^1$$

$$\mathbf{S}_K(x) = \frac{1}{2} 2^K\, K\, \mathbf{F}^0(x) + 2^K\, \mathbf{F}^1(x) + O(K)$$

$$\mathbf{F}^0(x) = \frac{1}{2} A_0 \mathbf{F}^0(2x), \qquad \text{for } 0 \le x < 1/2,$$

$$\mathbf{F}^0(x) = \frac{1}{2} A_0 V_2^0 + \frac{1}{2}\mathbf{F}^0(2x-1), \qquad \text{for } 1/2 \le x < 1;$$

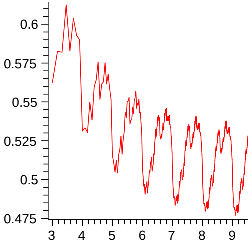
$$\mathbf{F}^1(x) = -\frac{1}{2}\mathbf{F}^0(x) + \frac{1}{2} A_0 \mathbf{F}^0(2x), \qquad \text{for } 0 \le x < 1/2,$$

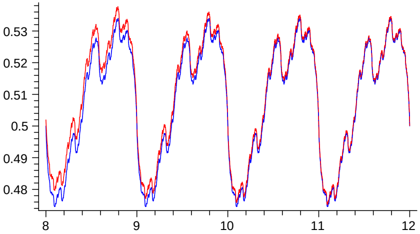
$$\mathbf{F}^1(x) = -\frac{1}{2}\mathbf{F}^0(x) + \frac{1}{2} A_0 V_2^1 + \frac{1}{2}\mathbf{F}^1(2x-1), \qquad \text{for } 1/2 \le x < 1$$

$\mathbf{F}^0(0) = 0$, $\mathbf{F}^0(1) = V_2^0$, $\mathbf{F}^1(0) = 0$, $\mathbf{F}^1(1) = V_2^1$. $\mathbf{F}^0(x) = xV_2^0$, $\mathbf{F}^1$ is not explicit.

$$\frac{1}{N}\sum_{n=1}^{N} E(n) \underset{N\to+\infty}{=} \frac{1}{4}\log_2 N + \frac{1}{4}\left(1 - \{t\} + 2^{3-\{t\}}\left(F_2^1(2^{\{t\}-1}) + F_3^1(2^{\{t\}-1}\right)\right) + O\left(\frac{\log N}{N}\right).$$

comparison between the (red) empirical and (blue) theoretical periodic functions





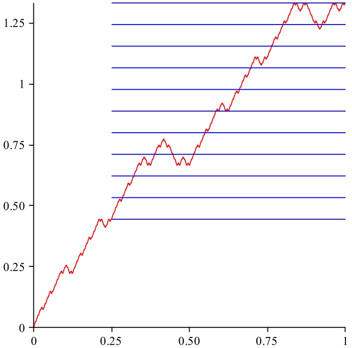
- Example Newman-Coquet sequence (Newman, 1969; Coquet, 1983)
  $u(n) = (-1)^{s_2(3n)}$ 4-rational sequence (changing the radix! $\pm\sqrt{3}$, $0 \to 3$, 0)

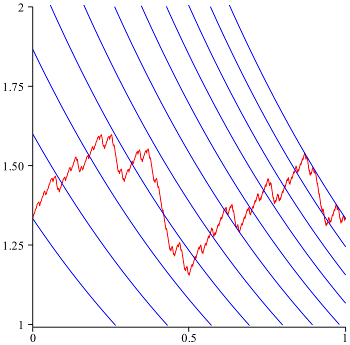
$$\sum_{n \le N} (-1)^{s_2(3n)} \underset{N \to +\infty}{=} N^{\log_4 3}\, 3^{1-\{t\}} F(4^{\{t\}-1}) + O(1),$$

$F = F_1 + F_2 + F_3$

$$\begin{cases} F_1(x) = \frac{1}{3} F_1(4x) + \frac{1}{3} F_2(4x) + \frac{1}{3} F_3(4x) + \frac{1}{3} F_1(4x-1), \\ F_2(x) = \frac{1}{3} F_2(4x-1) - \frac{1}{3} F_3(4x-1) + \frac{1}{3} F_1(4x-2) + \frac{1}{3} F_2(4x-2), \\ F_3(x) = \frac{1}{3} F_3(4x-2) + \frac{1}{3} F_1(4x-3) - \frac{1}{3} F_2(4x-3) + \frac{1}{3} F_3(4x-3), \end{cases}$$

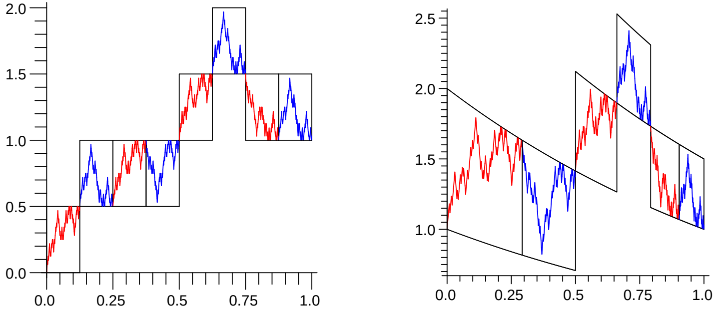
$F_1(0) = F_2(0) = F_3(0) = 0,\ F_1(1) = 2/3,\ F_2(1) = F_3(1) = 1/3.$





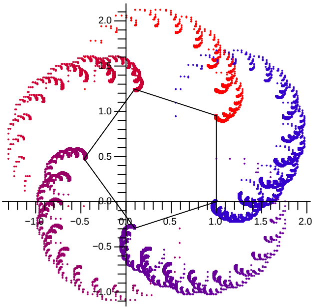
- Example Rudin-Shapiro sequence (Shapiro, 1951; Rudin, 1959; Brillhart and Carlitz, 1970)
  $u(n) = (-1)^{e_{2;11}(n)}$

$$\sum_{n \le N} u_n \underset{N \to +\infty}{=} \sqrt{N}\Phi(\log_4 N) + O(1)$$



- Periodicity versus pseudo-periodicity
  - $$\Phi(t) = \omega^{\lfloor t \rfloor} \rho^{1-\{t\}} F(B^{\{t\}-1})$$
  - Example Rosettes

$$A_0 = \begin{pmatrix} \cos\vartheta & 0 \\ 0 & \cos\vartheta \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & -\sin\vartheta \\ \sin\vartheta & 0 \end{pmatrix},$$



Context

- Exact expansion
  - Delange, 1975
    sum of digits
  - Allouche and Shallit, 2003
    extension
- Asymptotic expansion
  - Dumont *et alii*, 1989, 1990, 1999
    automata and substitutions
  - Flajolet *et alii*, 1994, 1994, 2008
    divide-and-conquer recurrences, Dirichlet series

$$U(s)(B^s\,\mathrm{I}_N - Q) = B^s \sum_{r=1}^{B-1} \frac{U_r}{r^s} + \sum_{r=1}^{B-1} \sum_{k=1}^{+\infty} (-1)^k \binom{s+k-1}{k} \left(\frac{r}{B}\right)^k U(s+k) A_r$$