Computing Closed-Form Solutions of Integrable Connections

Thomas Cluzeau

thomas.cluzeau@xlim.fr
http://www.ensil.unilim.fr/~cluzeau/

Joint work with M. Barkatou, C. El Bacha and J.-A. Weil

Algorithms Project's Seminar

INRIA Rocquencourt, March 26, 2012

Introducing example - G. Letac, W. Bryc (1)

$\diamond$ Problem in probability theory: find all probability distributions μ on real symmetric matrices of order n such that if X and Y are independent with the same distribution μ, then $X + Y = S$ and $S^{-1} X^2 S^{-1} = Z$ are independent.

$\diamond$ Under some restrictions, the problem can be reduced to (*Bryc-Letac'12*):

Find $y(x_1, \dots, x_n)$ such that

$$\forall j \in \{1, \dots, n\}, \quad \frac{\beta}{2}(j-n)\frac{\partial y}{\partial x_{j+1}} + \mathrm{Tr}(P_j \,\mathrm{Hess}(y)) = 0,$$

where β is *the Peirce constant* ($\beta \in \{1, 2, 4, 8, -2\}$), Hess the Hessian matrix and the P_j's are given symmetric matrices.

Introducing example - G. Letac, W. Bryc (2)

◇ Case $n = 2$:

$$\begin{cases} -\frac{\beta}{2}\,\frac{\partial y}{\partial x_2} + \frac{\partial^2 y}{\partial x_1^2} - x_2\,\frac{\partial^2 y}{\partial x_2^2} & = & 0 \\ 2\,\frac{\partial^2 y}{\partial x_1\,\partial x_2} + x_1\,\frac{\partial^2 y}{\partial x_2^2} & = & 0 \end{cases}$$

◇ Case $n = 3$:

$$\begin{cases} -\beta\,\frac{\partial y}{\partial x_2} + \frac{\partial^2 y}{\partial x_1^2} - x_2\,\frac{\partial^2 y}{\partial x_2^2} - 2\,x_3\,\frac{\partial^2 y}{\partial x_2\,\partial x_3} & = & 0 \\ -\frac{\beta}{2}\,\frac{\partial y}{\partial x_3} + 2\,\frac{\partial^2 y}{\partial x_1\,\partial x_2} + x_1\,\frac{\partial^2 y}{\partial x_2^2} - x_3\,\frac{\partial^2 y}{\partial x_3^2} & = & 0 \\ \frac{\partial^2 y}{\partial x_2^2} + 2\,\frac{\partial^2 y}{\partial x_1\,\partial x_3} + 2\,x_1\,\frac{\partial^2 y}{\partial x_2\,\partial x_3} + x_2\,\frac{\partial^2 y}{\partial x_3^2} & = & 0 \end{cases}$$

◇ Problem: compute "solutions" of such linear systems of PDEs

Contributions

◇ **Remark**: the latter systems are *D-finite* (*Chyzak-Salvy'98*)

◇ In this talk, we provide algorithms for computing:

- rational solutions
- hyperexponential solutions

of such D-finite linear systems of PDEs.

◇ Maple implementation available at

`http://www.ensil.unilim.fr/~cluzeau/PDS.html`

◇ Complexity analysis

Outline of the talk

1. D-finite linear systems of PDEs
2. Rational solutions
3. Hyperexponential solutions
4. Implementation
5. Conclusions

I

D-finite linear systems of PDEs

Notations and a definition

◇ C computable field of char. zero, $\overline{C}$ its algebraic closure

◇ $k = C(x_1, \ldots, x_m)$ and $K = \overline{C}(x_1, \ldots, x_m)$, $\partial_i = \partial/\partial x_i$

Definition

$\mathcal{U}$ universal differential extension of k containing all solutions of linear systems of PDEs over k (existence, e.g., *Kolchin'73*).

A linear system of PDEs is said to be *D*-finite if its solution space in $\mathcal{U}$ is of finite dimension over C.

◇ Algorithms to test if a given system is D-finite exist (*Chyzak-Salvy*'98 - Gröbner or Janet basis computations)
Implementation: OreModules (*Chyzak-Quadrat-Robertz*)

Integrable connections

Definition

Integrable connection over k of size n in m variables:

$$\begin{cases} \Delta_1 \, Y = 0 \quad \text{with} \quad \Delta_1 := \partial_1 \, I_n - A_1 \\ \quad\vdots \\ \Delta_m \, Y = 0 \quad \text{with} \quad \Delta_m := \partial_m \, I_n - A_m \end{cases}$$

where $A_i's \in \mathbb{M}_n(k)$ and the integrability conditions are satisfied:

$$\partial_i(A_j) - A_i \, A_j = \partial_j(A_i) - A_j \, A_i, \quad \forall \, i, j \in \{1, \ldots, m\}$$

$\diamond$ Every D-finite linear system of PDEs can be written as an integrable connection (*Chyzak-Salvy*'98), implementation in OreModules (*Chyzak-Quadrat-Robertz*)

Example: Bryc-Letac system for $n = 2$

$$\left\{ \begin{array}{ccc} -\frac{\beta}{2}\,\partial_2 y + \partial_1^2 y - x_2\,\partial_2^2 y & = & 0 \\ 2\,\partial_1\,\partial_2 y + x_1\,\partial_2^2 y & = & 0 \end{array} \right.$$

◇ Integrable connection over $\mathbb{Q}(\beta)$ of size 4 in 2 variables:

$$\partial_i\, Y - A_i\, Y = 0, \quad i = 1, 2, \quad \text{with}$$

$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\,x_1 \\ 0 & \frac{1}{2}\,\beta & 0 & x_2 \\ 0 & 0 & 0 & \frac{(-3-\beta)x_1}{{x_1}^2-4\,x_2} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{1}{2}\,x_1 \\ 0 & 0 & 0 & \frac{6+2\,\beta}{{x_1}^2-4\,x_2} \end{pmatrix}$$

◇ $Y = (y \quad \partial_2 y \quad \partial_1 y \quad \partial_2^2 y)^T$

Existing works

- Algorithmic studies of D-finite linear systems of PDEs:
  - *Chyzak'00*, *Oaku-Takayama-Tsai'01*: rational solutions of holonomic systems
  - *Li-Schwarz-Tsarev'03*: factorization, hyperexp. solutions
  - *Barkatou-Cluzeau-Weil'05*: factorization in char. p
  - *Wu'05*, *Li-Singer-Wu-Zheng'06*: Picard-Vessiot extensions, factorization, hyperexp. solutions over Laurent-Ore algebras
- Strategy of our work:
  - Consider integrable connections
  - Proceed recursively: benefit from algorithms for ordinary differential (OD) systems

II

Rational solutions

Rational solutions of OD systems (1)

⋄ C computable field of char. zero, $\overline{C}$ its algebraic closure, $k = C(x)$ and $K = \overline{C}(x)$

$$Y' = A\,Y, \quad A \in \mathbb{M}_n(k), \quad \mathrm{denom}(A) = \prod_{i=1}^{s} q_i(x)^{r_i+1}$$

⋄ Algorithm for computing rational solutions (for ex. *Barkatou'99*):

- Compute a universal denominator $Q = \prod_{i=1}^{s} q_i(x)^{m_i}$
- Compute polynomial solutions of $Z' = (A + (Q'/Q)\,I_n)\,Z$

Complexity estimate

$$Y' = A\,Y, \quad A = (a_{i,j})_{i,j} \in \mathbb{M}_n(k), \quad \mathrm{denom}(A) = \prod_{i=1}^{s} q_i(x)^{r_i+1}$$

$$d := \sum_{i=1}^{s} (r_i + 1)\,\deg(q_i)$$

$$r_\infty := \max\left(\max_{i,j} \left(1 + \deg(\mathrm{num}(a_{i,j})) - \deg(\mathrm{den}(a_{i,j}))\right), 0 \right)$$

◇ Arithmetic (operations in C) complexity estimate (*BCEW'12*):

- Universal denominator: simple form at q_i, integer roots of the *indicial polynomial*: $\mathcal{O}(n^5\, \max_i(r_i)\, d)$
- Polynomial solutions: degree bound (simple form at ∞), coefficients: $\mathcal{O}(n^5\, r_\infty^2 + n^3\, N^2)$

⇝ rational solutions of $Y' = A\,Y$: $\mathcal{O}(n^5\,(\max_i(r_i)\, d + r_\infty^2) + n^3\, N^2)$

◇ Main tool: simple form (arithm. compl. in *El Bacha's PhD'11*)

Rational solutions of integrable connections (1)

◇ $k = C(x_1, \dots, x_m)$, $K = \overline{C}(x_1, \dots, x_m)$

$$\left\{ \begin{array}{lll} \Delta_1 \, Y = 0 & \text{with} & \Delta_1 := \partial_1 \, I_n - A_1, \\ \quad \vdots & & \\ \Delta_m \, Y = 0 & \text{with} & \Delta_m := \partial_m \, I_n - A_m, \end{array} \right. \qquad A_i \in \mathbb{M}_n(k)$$

◇ **Notation**: $[A_1, \dots, A_m]$

Definition

Rational solution: vector $Y \in K^n$ such that $\Delta_i(Y) = 0$, $\forall \, i$.

◇ Recursive process:

- Compute $\mathcal{V} := \{Y \in K^n \, ; \, \Delta_1(Y) = 0\}$
- Reduce the size (m and n) of the problem

Rational solutions of integrable connections (2)

◇ $K_1 := \overline{C}(x_2, \dots, x_m)$, $K = K_1(x_1)$, $\mathcal{V} := \{Y \in K^n \,;\, \Delta_1(Y) = 0\}$

◇ $\mathcal{V}$ is a K_1-vector space stable under the action of each Δ_i

◇ A basis can be computed using an algorithm for OD systems and viewing $x_2, \dots, x_m$ as transcendental constants

Lemma

One can compute a non-singular matrix $P \in \mathbb{M}_n(K)$ such that, $\forall\, i$:

$$B_i := P^{-1}\,(A_i\,P - \partial_i(P)) = \begin{pmatrix} B_i^{11} & B_i^{12} \\ 0 & B_i^{22} \end{pmatrix}, \quad B_i^{11} \in \mathbb{M}_s(K).$$

Moreover, $B_1^{11} = 0$ and $\forall\, i = 2, \dots, m$, $B_i^{11} \in \mathbb{M}_s(K_1)$.

Rational solutions of integrable connections (3)

◇ $v_1, \dots, v_s$ K_1-basis of $\mathcal{V}$, $V = (v_1 \ \dots \ v_s) \in \mathbb{M}_{n \times s}(K)$

Theorem *(BCEW'12)*

$Y = V\,\Gamma \in K^n$ rat. sol. of $[A_1, \dots, A_m]$ iff $\Gamma \in K_1^s$ rat. sol. of

$$\begin{cases} \tilde{\Delta}_2\,\Gamma = 0 \quad \text{with} \quad \tilde{\Delta}_2 := \partial_2\, I_s - B_2^{11}, \\ \quad \vdots \\ \tilde{\Delta}_m\,\Gamma = 0 \quad \text{with} \quad \tilde{\Delta}_m := \partial_m\, I_s - B_m^{11}, \end{cases}$$

No more x_1!

⇝ Recursive algorithm (with efficient method for computing B_i^{11}'s)

◇ Complexity: worst case estimate (op. in k) ⇝ to be improved!

◇ Denominators: q irred. factor of the denom. of a rat. sol. such that $\partial_{i_0}(q) \neq 0 \Rightarrow q \,|\, \mathrm{denom}(A_{i_0})$ (*BCEW'12*)

III

Hyperexponential solutions

Exponential solutions of ordinary differential systems (1)

◇ C computable field of char. zero, $\overline{C}$ its algebraic closure, $k = C(x)$ and $K = \overline{C}(x)$

$$Y' = A\,Y, \quad A \in \mathbb{M}_n(k), \quad \mathrm{denom}(A) = \prod_{i=1}^{s} q_i(x)^{r_i+1}$$

Definition

Exponential solution: $\exp(\int f \, \mathrm{d}x)\, z$, where $f \in K$ and $z \in K^n$.

◇ Algorithm for computing exponential solutions (*Pfluegel'01*):

- Compute the non-ramified local exponential parts at each sing.
- For each combination, compute polynomial solutions

◇ Bottlenecks: large number of comb. & computations in algebraic extensions of C of large degree

Exponential parts and complexity estimate

$$Y' = A\,Y, \quad A = \frac{1}{x^{r+1}}\,(A_0 + A_1\,x + A_2\,x^2 + \cdots),\ r \in \mathbb{N},\ A_i \in \mathbb{M}_n(\overline{C})$$

Definition

Non-ramified local exponential part at $x = 0$: polynomial $\tilde{f}$ in $1/x$

$$\tilde{f} = \frac{\alpha_{p+1}}{x^{p+1}} + \frac{\alpha_p}{x^p} + \cdots + \frac{\alpha_1}{x},$$

where $0 \leq p \leq r$ and $\alpha_i' s \in \overline{C}$ such that there exists a formal local solution of the system of the form $\exp(\int \tilde{f}\, \mathrm{d}x)\, \tilde{z}$, where $\tilde{z}$ is a vector of formal power series in x.

◇ Arithmetic cost (*BCEW'12*): $\mathcal{O}(n^5\, r^3\, \min(n, r))$ op. in an alg. ext. of C of degree $\leq n$ (super-reduction, *Barkatou-Pfluegel'09*)

Complexity estimate

$$Y' = A\,Y, \quad A = (a_{i,j})_{i,j} \in \mathbb{M}_n(k), \quad \mathrm{denom}(A) = \prod_{i=1}^{s} q_i(x)^{r_i+1}$$

$$d := \sum_{i=1}^{s} (r_i + 1)\,\deg(q_i)$$

$$r_\infty := \max\left(\max_{i,j}\left(1 + \deg(\mathrm{num}(a_{i,j})) - \deg(\mathrm{den}(a_{i,j}))\right), 0\right)$$

$\rightsquigarrow$ Exponential solutions of $Y' = A\,Y$ (*BCEW'12*):

- $\mathcal{O}(n^5\,(\max_i(r_i)^2\,d\,\sum_i \min(n, r_i) + r_\infty^3\,\min(n, r_\infty)))$ op. in an alg. ext. of C of degree $\leq n$

- $\mathcal{O}(n^{\delta+3}\,N^2)$ op. in an alg. ext. of C of degree $\leq n^\delta\,\delta!$

(δ: number of singularities, N: degree bound for all the computed polynomial solutions)

Hyperexponential solutions of integrable connections (1)

$$\left\{ \begin{array}{lll} \Delta_1 Y = 0 & \text{with} & \Delta_1 := \partial_1 I_n - A_1, \\ \quad \vdots & & \\ \Delta_m Y = 0 & \text{with} & \Delta_m := \partial_m I_n - A_m, \end{array} \right. \qquad A_i \in \mathbb{M}_n(C(x_1, \ldots, x_m))$$

◇ $K = \overline{C}(x_1, \ldots, x_m)$

Definition

L differential extension of K having the same field of constants.
⊙ $u \neq 0 \in L$ hyperexponential over K: $\forall\, i,\ f_i := \partial_i(u)/u \in K$.
⊙ hyperexponential solution: solution $u\,z$ with u hyperexponential over K and $z \in K^n$.

◇ u hyperexponential over $K \Rightarrow \partial_j(f_i) = \partial_i(f_j),\ \forall\, i,\, j$

◇ $u\,z$ hyperexp. sol. of $[A_1, \ldots, A_m]$
$\Rightarrow z$ rat. sol. of $[A_1 - f_1\, I_n, \ldots, A_m - f_m\, I_n]$

Hyperexponential solutions of integrable connections (2)

◇ Recursive algorithm as for rational solutions

- Exp. sol. of $Y' = A_1\, Y$ computed with algorithm for OD systems. Let $u\, z$ be such a solution
- $f_i := \partial_i(u)/u \in K$ and $\Delta_{i,u} := \partial_i - (A_i - f_i\, I_n)$
- $w_1, \ldots, w_s$ basis of $\mathcal{W}_u = \{w \in K^n\, ;\, \Delta_{1,u}(w) = 0\}$, complete it into a basis of $K^n \rightsquigarrow$ matrix $P = (W_u \quad \tilde{W})$

Theorem (BCEW'12)

$Y = u\, W_u\, \Gamma_u$ hyperexp. sol. of $[A_1, \ldots, A_m]$ iff Γ_u hyperexp. sol. of $[B_2^{11}, \ldots, B_m^{11}]$ where $B_i = P^{-1}\,((A_i - f_i\, I_n)\, P - \partial_i(P))$ and $B_i^{11} \in \mathbb{M}_s(K_1)$ denotes the first $s \times s$ submatrix of B_i.

◇ Complexity: worst case estimate $\rightsquigarrow$ to be improved

◇ Discard local exp. parts involving non-rat. functions of x_j's, $j \neq 1$

IV

Implementation

Maple package IntegrableConnections

◇ Algorithms are implemented in a Maple package called IntegrableConnections

- Available with some examples at

  `http://www.ensil.unilim.fr/~cluzeau/PDS.html`

- Main procedures: *RationalSolutions* (& *Eigenring*), *HyperexponentialSolutions*
- Some adaptations of Isolde code (*Barkatou-Pfluegel*)

Demo.

V

Conclusions

Contributions and Perspectives

◇ Summary of the contributions:

- Complexity estimates for computing rat. and exp. solutions of OD systems (in the literature of OD systems, *Grigoriev'90*)
- Algorithms for computing rational and hyperexponential solutions of integrable connections
- Implementation available (IntegrableConnections)

◇ Perspectives:

- Precise complexity analysis of algorithms for integrable connections
- Algorithms for other types of solutions and factorization