LLL-reducing in quasi-linear time

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Euclidean lattices

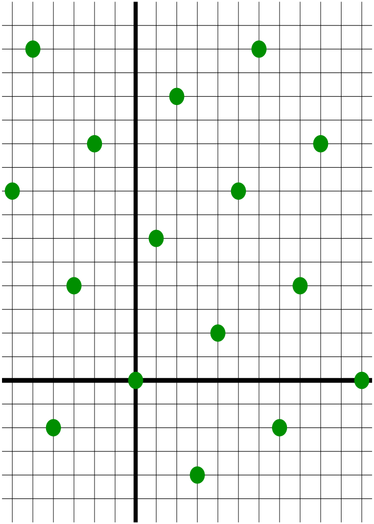
Lattice $\equiv$ discrete subgroup of $\mathbb{R}^n$

$\equiv \quad \{\sum_{i \leq n} x_i \mathbf{b}_i : x_i \in \mathbb{Z}\}$

If the $\mathbf{b}_i$'s are linearly independent, they are called a basis.

Bases are not unique, but they can be obtained from each other by integer transforms of determinant ± 1:

$$\begin{bmatrix} -2 & 1 \\ 10 & 6 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$



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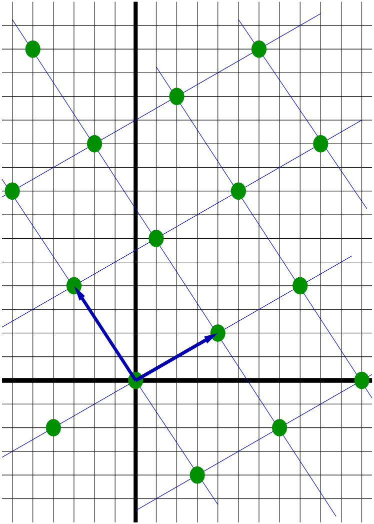

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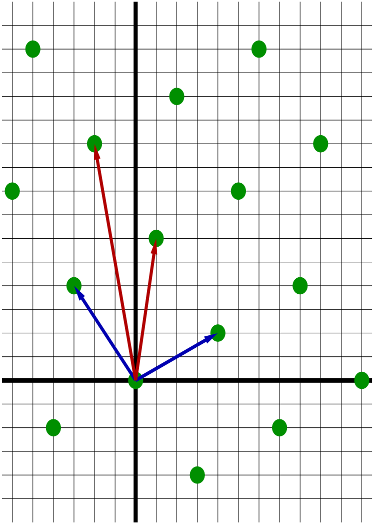

Lattice reduction: a representation paradigm

Lattice reduction: Start from an arbitrary basis, and improve the norms/orthogonality of its vectors.

What for?

- Shorter vectors $\Rightarrow$ less space.
- Reduced bases provide intrinsic information about the lattice.
- Reduced bases are easier to compute with.

Lattice reduction as a matrix problem:

Given $B \in \mathbb{R}^{n \times n}$ full-rank, find $U \in GL_n(\mathbb{Z})$ s.t.

BU small and/or with a "nice" QR-factor R.

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Why do we care about lattices?

- Computer algebra: factorisation of rational polynomials.
- Cryptography: cryptanalyses of variants of RSA.
- Communications theory: MIMO, GPS, error correcting codes.
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Lattices tend to pop out every time one wants to use linear algebra but is restricted to discrete transformations.

The LLL reduction [Lenstra-Lenstra-Lovász'82]

Let $\delta \in (1/4, 1)$. A basis $B = (\mathbf{b}_i)_{i \leq n} \in \mathbb{R}^{n \times n}$ with QR-factorisation $B = QR$ is said LLL-reduced if:

- $\forall i, j : \; |r_{i,j}| \leq r_{i,i}/2$ [size-reduction]
- $\forall i : \; \delta \cdot r_{i,i}^2 \leq r_{i,i+1}^2 + r_{i+1,i+1}^2$ [Lovász' condition].

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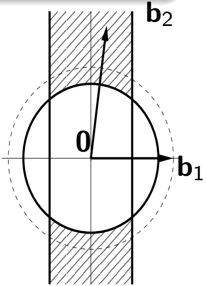
The $r_{i,i}$'s can't drop too fast: $r_{i+1,i+1} \geq \sqrt{\delta - \frac{1}{4}}\, r_{i,i}$.
$\prod_i \|\mathbf{b}_i\| \leq 2^{\mathcal{O}(n^2)} \cdot \det(L)$.

$\det(L) := \det(\mathbf{b}_i)_i$ is a lattice invariant.
$\delta < 1$ is crucial to get polynomial-time complexity.



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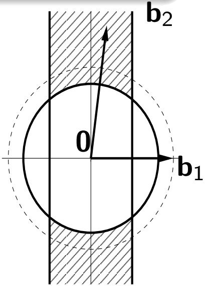

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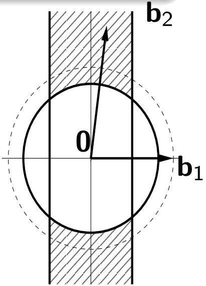

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Complexity bounds

Input: $B \in \mathbb{Z}^{n \times n}$ of full rank, with $\max \|\mathbf{b}_i\| \leq 2^\beta$.

LeLeLo'82	LLL/L^3	$n^{5+\varepsilon}\beta^{2+\varepsilon}$
Kaltofen'83		$n^5\beta^2(n+\beta)^\varepsilon$
Schnorr'87		$n^4\beta(n+\beta)^{1+\varepsilon}$
Nguyen-S.'05	L^2	$n^{4+\varepsilon}\beta(n+\beta)$
Morel-S.-Villard'08	H-LLL	same bound, simpler proof

Can we do better with respect to β?

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Quasi-linear LLL-reduction

- Yap'92, Schönhage'91: $\beta^{1+\varepsilon}$ for $n = 2$.
- Eisenbrand-Rote'01: $\beta^{1+\varepsilon}$ for fixed any n.

Our result

We give an algorithm, called $\widetilde{L}^1$, that computes "somewhat" LLL-reduced bases in time $\mathcal{O}(n^{5+\varepsilon}\beta + n^{\omega+1+\varepsilon}\beta^{1+\varepsilon})$.

- n^ω: cost of matrix mult. in dimension n.
- For fixed n: $\mathcal{O}(\mathcal{M}(\beta)\log\beta)$, where $\mathcal{M}(\cdot)$ is for integer mult.
- Same total degree as before.

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Plan of the talk

1. **Wishful thinking.**
2. Reducing by deforming.
3. Reducing by truncating.
4. The $\widetilde{L}^1$ algorithm.

A gcd analogy

Euclid's algorithm for computing $\gcd(r_0, r_1)$:

$i := 1$. While $r_i \neq 0$:
Compute $q_i := \lfloor r_{i-1}/r_i \rfloor$, $r_{i+1} := r_{i-1} - q_i r_i$.
Output r_{i-1}.

Vectorial interpretation:

$$\begin{pmatrix} r_i \\ r_{i+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -q_i \end{pmatrix} \cdot \begin{pmatrix} r_{i-1} \\ r_i \end{pmatrix} = \prod_{j=i}^{1} \begin{pmatrix} 0 & 1 \\ 1 & -q_j \end{pmatrix} \cdot \begin{pmatrix} r_0 \\ r_1 \end{pmatrix}$$

LLL as a gcd: Given B_i, find U_i s.t. $B_i U_i$ is closer to reduced.

- L^3: Compute r_{i-1}/r_i exactly before rounding it.
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Towards a quasi-linear time gcd algorithm

Euclid computes remainders $(r_i)_i$ and quotients $(q_i)_i$.

- Assume $r_0 \approx r_1 \approx 2^\beta$.
- Writing down all the r_i's costs $\mathcal{O}(\beta^2)$.

Lehmer'38

If $\frac{|r_0 - \bar{r}_0|}{|r_0|}, \frac{|r_1 - \bar{r}_1|}{|r_1|} \leq 2^{-2\ell}$, then $(q_i)_i$ and $(\bar{q}_i)_i$ share their first ℓ bits.

- Do not compute the q_i's using and updating the lengthy r_i's: Use the shorter $\bar{r}_i$'s instead!
- When the relevant bits of the q_i's are known, apply them to (r_0, r_1)... and apply Lehmer again.
- Knuth'70, Schönhage'71: Do this recursively!

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The Knuth-Schönhage gcd algorithm

To compute the first ℓ quotient bits of r_0, r_1 of bit-sizes 2ℓ:

1. Take the first ℓ bits of r_0 and r_1.
2. Recursively get the first $\ell/2$ quotient bits.
3. Apply the quotients to r_0, r_1, to get r_0', r_1'.
4. Take the first ℓ bits of r_0' and r_1'.
5. Recursively get the first $\ell/2$ quotient bits.

- Applying the quotients: multiply a $\mathcal{O}(\ell)$-bit 2×2 matrix to a $\mathcal{O}(\ell)$-bit vector.
- Cost: $C_\ell = 2C_{\ell/2} + \mathcal{O}(\mathcal{M}(\ell)) = \mathcal{O}(\mathcal{M}(\ell) \log \ell)$.
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What about doing it for LLL?

To compute the "first" ℓ bits of U reducing B:

1. Take the first ℓ bits of each b_{ij}.
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- What is a "quotient" here?
- How to control the bit-size of a unimodular matrix?
- Can we truncate "remainders", i.e., lattice bases?
- How to handle multidimensionality / unbalanced magnitudes?

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From reduced to reduced

- If B is arbitrary, then a reducing U can be huge (Cramer :-().
- If B is reduced, any U such that BU is reduced is bounded.

Let B be reduced with R-factor R, and U s.t. BU is reduced. Then:

$$\forall i, j : |u_{ij}| \leq 2^{\mathcal{O}(n)} \cdot r_{jj}/r_{ii}.$$

- If B is reduced, the r_{ii}'s can't decrease fast.
- Assuming they don't increase, we get $\max |u_{ij}| \leq 2^{\mathcal{O}(n)}$.

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From reduced to deformed to reduced

- Start from something reduced, deform it a bit, and reduce it!
- The Belabas-van Hoeij-Novocin deformation:

$$B \mapsto \mathrm{diag}(2^\ell, 1, \ldots, 1) \cdot B = \sigma_\ell B.$$

- The r_{ii}'s cannot decrease.
- Their product increases by a factor 2^ℓ.

Let $\ell \geq 0$, B be reduced with R-factor R, and U s.t. $\sigma_\ell BU$ is reduced. Then:

$$\forall i, j : |u_{ij}| \leq 2^{\ell + \mathcal{O}(n)} \cdot r_{jj} / r_{ii}.$$

$\longrightarrow$ If B is "balanced", each u_{ij} has at most $\ell + \mathcal{O}(n)$ bits.

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$\longrightarrow$ If B is "balanced", each u_{ij} has at most $\ell + \mathcal{O}(n)$ bits.

Lift-reducing suffices for reducing

Assume $B \in \mathbb{Z}^{n \times n}$ is upper triangular.

$$\begin{bmatrix} b_{1,1} & \dots & b_{1,n-2} & b_{1,n-1} & b_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & b_{n-2,n-2} & b_{n-2,n-1} & b_{n-2,n} \\ 0 & \dots & 0 & b_{n-1,n-1} & b_{n-1,n} \\ 0 & \dots & 0 & 0 & b_{n,n} \end{bmatrix}$$

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Bottom right 1×1 submatrix is reduced.

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Scale down row $n-1$ so that bottom-right 2×2 submatrix is reduced: $\ell \approx \log b_{n-1,n-1}$.

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Propagate the transformations to the first $n-2$ coordinates, and reduce wrt the diagonal coefficients.

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Keep going.

Lift-reducing in quasi-linear time suffices

- McCurley-Hafner'91:
  $H = \mathrm{HNF}(B)$ can be computed in time $\mathcal{O}(n^{\omega+1+\varepsilon}\beta^{1+\varepsilon})$.
- Cost of the lifts:
  $$\mathcal{P}oly(n) \cdot \left(\widetilde{O}(\log h_{n,n}) + \widetilde{O}(\log h_{n-1,n-1}) + \ldots\right)$$
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  (in fact, we do a bit better than that)
- Cost of the propagations bounded using the smallness of the transforms: $\mathcal{O}(n^{\omega+1+\varepsilon}(\beta^{1+\varepsilon} + n))$.

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Where are we now?

- LLL-reduction $\longrightarrow$ sequence of Lift-reductions.
- We are to lift-reduce in quasi-linear time.
- More precisely: given ℓ and B reduced, we will find U unimodular such that $\sigma_\ell BU$ is reduced, in time $\widetilde{O}(\ell)$.
- This is independent from the bit-size of B.

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2. Reducing by deforming.
3. **Reducing by truncating.**
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$$\underset{\text{Reduced}}{\begin{bmatrix} 1 & 2^{53}+2^{-1}+2^{-25} \\ 2^{-10} & -2^{63} \end{bmatrix}} \quad \Longrightarrow \quad \underset{\text{Not reduced}}{\begin{bmatrix} 1 & 2^{53}+1 \\ 2^{-10} & -2^{63} \end{bmatrix}}$$

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If $B \in \mathbb{Z}^{n\times n}$, we may need all the bits to decide.

If $B \in \mathbb{R}^{n\times n}$, we may not even be able to tell!

Sensitivity of the R-factor

- Take $B \in \mathbb{R}^{n \times n}$ full-rank, with $B = QR$.
- Apply a columnwise perturbation ΔB, i.e., $\max_i \frac{\|\Delta \mathbf{b}_i\|}{\|\mathbf{b}_i\|} \leq \varepsilon$.
- If ε is very small, then $B + \Delta B$ is full-rank and:

$$B + \Delta B = (Q + \Delta Q)(R + \Delta R).$$

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Let $\mathrm{cond}(R) = \| |R||R^{-1}| \|$. If $\mathrm{cond}(R) \cdot \varepsilon \lesssim 1$, then:
$B + \Delta B$ is full-rank and $\max \frac{\|\Delta \mathbf{r}_i\|}{\|\mathbf{r}_i\|} \lesssim \mathrm{cond}(R) \cdot \varepsilon$.

Furthermore, if B is LLL-reduced, then $\mathrm{cond}(R) = 2^{\mathcal{O}(n)}$.

Fixing the LLL-reduction

- We would like the reduction to resist perturbations.
- The bound on $\|\Delta \mathbf{r}_j\|$ is proportional to $\|\mathbf{r}_j\|$.
- By reducedness, $1 \leq \frac{\|\mathbf{r}_j\|}{r_{j,j}} \leq 2^{\mathcal{O}(n)}$.

$\Rightarrow$ $r_{i,j}$ should be related to $r_{j,j}$ instead of (only) $r_{i,i}$.

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Let $\Xi = (\delta, \eta, \theta)$ with $\eta \in (1/2, 1)$, $\theta > 0$ and $\delta \in (\eta^2, 1)$.
A basis $B \in \mathbb{R}^{n \times n}$ with R-factor R is said Ξ-reduced if:

- $\forall i, j : \ |r_{i,j}| \leq \eta \cdot r_{i,i} + \theta \cdot r_{j,j}$ [Modified size-reduction]
- $\forall i : \ \delta \cdot r_{i,i}^2 \leq r_{i,i+1}^2 + r_{i+1,i+1}^2$.

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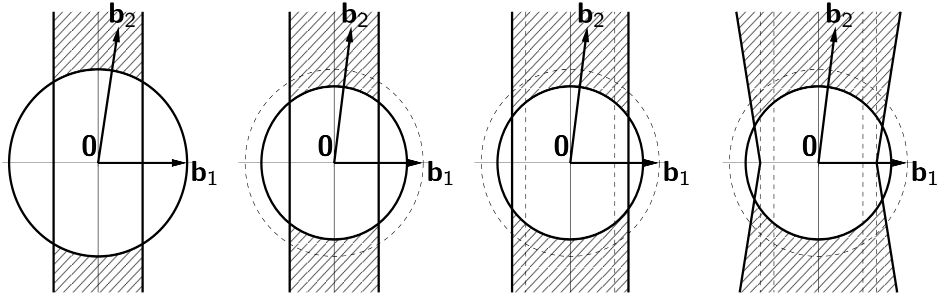
If B is balanced, this is the same as before.

The LLL-reductions, graphically



$(1, 1/2, 0)$	$(\delta, 1/2, 0)$	$(\delta, \eta, 0)$	(δ, η, θ)
Hermite	LLL'82	Schnorr'88	Chang-S-Villard'11

Properties of the new reduction

The new reduction is perturbation-friendly:

- We still have $\mathrm{cond}(R) = 2^{\mathcal{O}(n)}$ for Ξ-reduced bases.
- If B is reduced and $\max \frac{\|\Delta \mathbf{b}_i\|}{\|\mathbf{b}_i\|} \leq 2^{-\Omega(n)}$,
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The popular properties of LLL-reduction still hold:

- Computable in polynomial time.
- B reduced $\implies \prod \|\mathbf{b}_i\| \leq 2^{\mathcal{O}(n^2)} \cdot |\det(\mathbf{b}_i)_i|$.

Deformations and truncations are compatible

- B and $\sigma_\ell BU$ reduced $\implies$ U small.
- B reduced $\implies$ $B + \Delta B$ reduced.

Let $\ell \geq 0$, B be reduced and ΔB s.t. $\max \frac{\|\Delta \mathbf{b}_i\|}{\|\mathbf{b}_i\|} \leq 2^{-\ell-\Omega(n)}$.
If $\sigma_\ell(B + \Delta B)U$ is reduced, then so is $\sigma_\ell BU$...
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Overview of $\widetilde{L}^1$

- $\widetilde{L}^1$: HNF and n calls to `Lift-`$\widetilde{L}^1$.
- If B is reduced and $\ell \geq 0$, `Lift-`$\widetilde{L}^1$ computes U unimodular such that $\sigma_\ell BU$ is reduced, in time $\mathcal{P}oly(n) \cdot \widetilde{O}(\ell)$.

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A first attempt for Lift-$\widetilde{L}^1$

Inputs: B reduced, lifting target ℓ.
Output: U unimodular such that $\sigma_\ell BU$ reduced.

- Keep the $\ell/2 + O(n)$ top-most bits of B.
- Recursively compute U_1 s.t. $\sigma_{\ell/2} BU_1$ reduced.
- Apply U_1 to $\sigma_{\ell/2} B$ and keep the $\ell/2 + O(n)$ top-most bits.
- Recursively compute U_2 s.t. $\sigma_{\ell/2}(\sigma_{\ell/2} BU_1)U_2$ is reduced.
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Some additional difficulties

1. Keep the $\ell/2 + O(n)$ top-most bits of B.
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4. Recursively compute U_2 s.t. $\sigma_{\ell/2}(\sigma_{\ell/2} B U_1) U_2$ is reduced.
5. Return $U_1 \cdot U_2$.

- What do we do at the recursion leaves?
- Every time we truncate, we may loosen the reduction factors...
- How do we compute $B \cdot U_1$ and $U_1 \cdot U_2$ efficiently?

Strengthening the reducedness of a basis (Morel-S-Villard)

Problem: Suppose we have a Ξ-reduced basis.
How do we Ξ'-reduce it, for $\Xi' > \Xi$?

- Truncate, reduce, output the obtained U.
- This takes time $\mathcal{O}(n^{6+\varepsilon})$ when the r_{ii}'s are balanced.

Otherwise, u_{ij} can be as large as r_{jj}/r_{ii} ...

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Reducedness strengthening

- Used for the recursion leaves.
- Used for re-strengthening the reduction factors, loosened by the truncations.
- Returns (U, S) s.t.:
  - $B(SUS^{-1})$ is reduced,
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Bounding the cost of $\texttt{Lift-}\widetilde{\mathtt{L}}^1$

1. Keep the $\ell/2 + O(n)$ top-most bits of B.
2. Recursively compute U_1 s.t. $\sigma_{\ell/2} B U_1$ reduced.
3. Apply U_1 to $\sigma_{\ell/2} B$ and keep the $\ell/2 + O(n)$ top-most bits.
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5. Return $U_1 \cdot U_2$.

New representations for bases and transforms:

- Easy if assuming all handled bases are "balanced". Else...
- An ℓ-lifing U is stored as (U', D) with $U = DU'D^{-1}$, $\max |u'_{ij}| \leq 2^{\ell + \mathcal{O}(n)}$ and D p-of-2 diagonal.
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Final hassle:

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Sanitizing the transforms

Assume B and $\sigma_\ell BU$ are reduced with $\ell \geq 0$ and U unimodular. Let ΔU s.t. $|\Delta u_{ij}| \leq 2^{-\Omega(\ell+n)} r_{jj}/r_{ii}$, then:

$$U + \Delta U \text{ unimodular} \quad \text{and} \quad \sigma_\ell B(U + \Delta U) \text{ reduced.}$$

- A lift-reducing U may be large, but its bit-size can be made small.
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- This generalizes Knuth-Schönhage and Schönhage-Yap to arbitrary dimensions.
- Three ingredients: deforming, truncating, Knuth-Schönhage.

1. Can we do better wrt n? [Schönhage'84, Storjohann'96, etc]
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