On the Structure of Compatible Rational Functions

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Outline

1. Bivariate compatible rational functions
2. General case
3. A structure theorem
4. Multiplicative decomposition of hyperexp.-hypergeom. elements
5. Applications
6. Algebraic dependence of hyperexp.-hypergeom. elements

Note. $\mathbb{F}$ *is an algebraically closed field of characteristic zero.*

Bivariate case 1: differential and difference

Let

$$\begin{array}{rcl} \delta : \; \mathbb{F}(t,x) & \rightarrow & \mathbb{F}(t,x) \\ f(t,x) & \mapsto & \frac{\partial f(t,x)}{\partial t} \end{array} \quad \text{and} \quad \begin{array}{rcl} \sigma : \; \mathbb{F}(t,x) & \rightarrow & \mathbb{F}(t,x) \\ f(t,x) & \mapsto & f(t,x+1) \end{array}$$

A first-order system

$$\begin{cases} \delta(z) = u\,z, \\ \sigma(z) = v\,z, \end{cases} \quad \text{where } u, v \in \mathbb{F}(t,x) \text{ with } v \neq 0,$$

has a nonzero solution iff the following compatibility condition (CC) holds:

$$\frac{\delta(v)}{v} = \sigma(u) - u.$$

$$\begin{cases} \sigma \circ \delta(z) & = & \sigma(u)\,v\,z \\ \delta \circ \sigma(z) & = & (\delta(v) + uv)\,z \end{cases} \quad \Longrightarrow \quad \text{the CC.}$$

Definition. $u, v \in \mathbb{F}(t,x)$ are compatible w.r.t. $\{\delta,\ \sigma\}$ if $v \neq 0$ and

$$\frac{\delta(v)}{v} = \sigma(u) - u.$$

Lemma. (R. Feng, M. Singer, and M. Wu, 2010)

$u, v \in \mathbb{F}(t,x)$ are compatible w.r.t. $\{\delta,\ \sigma\}$ iff $\exists\, f \in \mathbb{F}(t,x),\ \alpha, \beta \in \mathbb{F}(t),\ \lambda \in \mathbb{F}(x)$ s.t.

$$u = \frac{\delta(f(t,x))}{f(t,x)} + x\frac{\delta(\alpha(t))}{\alpha(t)} + \beta(t)$$

and

$$v = \frac{\sigma(f(t,x))}{f(t,x)}\,\alpha(t)\,\lambda(x).$$

Corollary. A solution of $\{\delta(z) = u\,z,\ \sigma(z) = v\,z\}$ can be written as

$$f(t,x)\,\alpha(t)^x\ \exp\left(\int \beta(t)dt\right)\ T(x),$$

where $\delta(T) = 0$ and $\sigma(T) = \lambda\,T$.

Bivariate case 2: differential and q-difference

Let

$$\begin{array}{rcl} \delta: \mathbb{F}(t,y) & \to & \mathbb{F}(t,y) \\ f(t,y) & \mapsto & \frac{\partial f(t,y)}{\partial t} \end{array} \quad \text{and} \quad \begin{array}{rcl} \tau: \mathbb{F}(t,y) & \to & \mathbb{F}(t,y) \\ f(t,y) & \mapsto & f(t, q\, y), \end{array}$$

where $q \in \mathbb{F}$ is not a root of unity.

A first-order system:

$$\begin{cases} \delta(z) = u\, z, \\ \tau(z) = w\, z, \end{cases} \qquad u, w \in \mathbb{F}(t,y) \text{ with } w \neq 0,$$

has a nonzero solution iff the following CC holds:

$$\frac{\delta(w)}{w} = \tau(u) - u.$$

Definition. $u, w \in \mathbb{F}(t, y)$ are compatible w.r.t. $\{\delta, \tau\}$ if $w \neq 0$ and

$$\frac{\delta(w)}{w} = \tau(u) - u.$$

Lemma. (a q-analogue of FSW Lemma)

$u, w \in \mathbb{F}(t, y)$ are compatible w.r.t. $\{\delta, \tau\}$ iff $\exists\, f \in \mathbb{F}(t, y)$, $\beta \in \mathbb{F}(t)$, $\mu \in \mathbb{F}(y)$ s.t.

$$u = \frac{\delta(f(t, y))}{f(t, y)} + \beta(t) \quad \text{and} \quad v = \frac{\tau(f(t, y))}{f(t, y)}\, \mu(y).$$

Corollary. A solution of $\{\delta(z) = u\, z,\ \tau(z) = w\, z\}$ can be written as

$$f(t, y) \exp\left(\int \beta(t) dt\right) Q(y),$$

where $\delta(Q) = 0$ and $\tau(Q) = \mu\, Q$.

Comparison

Differential and difference.

Fact. For $g \in \mathbb{F}(t, x)$,

$$\underline{\sigma(g) - g \in \mathbb{F}(t) \quad \Leftrightarrow \quad g \in \mathbb{F}(t)[x] \text{ and } \deg_x g \leq 1.}$$

$$\Downarrow$$

$$u = \frac{\delta(f(t,x))}{f(t,x)} + x\frac{\delta(\alpha(t))}{\alpha(t)} + \beta(t) \quad \text{and} \quad v = \frac{\sigma(f(t,x))}{f(t,x)}\alpha(t)\,\lambda(x).$$

Differential and q-difference.

Fact. For $g \in \mathbb{F}(t, y)$,

$$\underline{\tau(g) - g \in \mathbb{F}(t) \quad \Leftrightarrow \quad g \in \mathbb{F}(t).}$$

$$\Downarrow$$

$$u = \frac{\delta(f(t,y))}{f(t,y)} + \beta(t) \quad \text{and} \quad w = \frac{\tau(f(t,y))}{f(t,y)}\mu(y).$$

Bivariate Case III: difference and q-difference

Let

$$\begin{array}{rcl} \sigma: \ \mathbb{F}(x,y) & \to & \mathbb{F}(x,y) \\ f(x,y) & \mapsto & f(x+1,y) \end{array} \quad \text{and} \quad \begin{array}{rcl} \tau: \ \mathbb{F}(x,y) & \to & \mathbb{F}(x,y) \\ f(x,y) & \mapsto & f(x,q\,y), \end{array}$$

where $q \in \mathbb{F}$ is not a root of unity.

For $v, w \in \mathbb{F}(x,y)$ with $vw \neq 0$, the system:

$$\begin{cases} \sigma(z) = v\,z \\ \tau(z) = w\,z \end{cases}$$

has a nonzero solution iff the following CC holds:

$$\frac{\sigma(w)}{w} = \frac{\tau(v)}{v}.$$

Definition. $v, w \in \mathbb{F}(t,x)$ are compatible w.r.t. $\{\sigma,\tau\}$ if $vw \neq 0$ and

$$\frac{\sigma(w)}{w} = \frac{\tau(v)}{v}.$$

Lemma. v and w are compatible w.r.t. $\{\sigma,\tau\}$ iff there exist $f \in \mathbb{F}(x,y)$, $\lambda \in \mathbb{F}(x)$, $\mu \in \mathbb{F}(y)$ s.t.

$$v = \frac{\sigma(f(x,y))}{f(x,y)}\,\lambda(x) \quad \text{and} \quad w = \frac{\tau(f(x,y))}{f(x,y)}\,\mu(y).$$

Corollary. A solution of

$$\begin{cases} \sigma(z) = v\,z \\ \\ \tau(z) = w\,z \end{cases}$$

can be written as

$$c\,f(x,y)\,T(x)\,Q(y),$$

where $c \in \mathbb{F}$, $\sigma(T) = \lambda\,T$ and $\tau(Q) = \mu\,Q$.

Survey on mixed cases

Operators	CC	Rational solutions
$\delta = \frac{\partial}{\partial t}$ $\sigma : x \mapsto x+1$	$\frac{\delta(v)}{v} = \sigma(u) - u$	$u = \frac{\delta(f)}{f} + x\frac{\delta(\alpha)}{\alpha} + \beta,\ v = \frac{\sigma(f)}{f}\,\alpha\,\lambda$ where $f \in \mathbb{F}(t,x)$, $\alpha, \beta \in \mathbb{F}(t)$, $\lambda \in \mathbb{F}(x)$
$\delta = \frac{\partial}{\partial t}$, $\tau : y \mapsto q\,y$	$\frac{\delta(w)}{w} = \tau(u) - u$	$u = \frac{\delta(f)}{f} + \beta,\ w = \frac{\tau(f)}{f}\,\mu$ where $f \in \mathbb{F}(t,y)$, $\beta \in \mathbb{F}(t)$, $\mu \in \mathbb{F}(y)$
$\sigma : x \mapsto x+1$, $\tau : y \mapsto q\,y$	$\frac{\sigma(w)}{w} = \frac{\tau(v)}{v}$	$v = \frac{\sigma(f)}{f}\,\lambda,\ w = \frac{\tau(f)}{f}\,\mu$, where $f \in \mathbb{F}(x,y)$, $\lambda \in \mathbb{F}(x)$, $\mu \in \mathbb{F}(y)$.

General case

Let $\mathbf{t} = (t_1, \ldots, t_\ell)$, $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_n)$.

For $i = 1, \ldots, \ell$,

$$\begin{aligned} \delta_i : \ \mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y}) &\longrightarrow \mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y}) \\ f &\mapsto \frac{\partial f}{\partial t_i}. \end{aligned}$$

For $j = 1, \ldots, m$,

$$\begin{aligned} \sigma_j : \quad \mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y}) &\longrightarrow \mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y}) \\ f(\mathbf{t}, x_1, \ldots, x_m, \mathbf{y}) &\mapsto f(\mathbf{t}, x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_m, \mathbf{y}). \end{aligned}$$

For $k = 1, \ldots, n$,

$$\begin{aligned} \tau_k : \quad \mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y}) &\longrightarrow \mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y}) \\ f(\mathbf{t}, \mathbf{x}, y_1, \ldots, y_n) &\mapsto f(\mathbf{t}, \mathbf{x}, y_1, \ldots, y_{k-1}, q_k y_k, y_{k+1}, \ldots, y_n), \end{aligned}$$

where $q_1, \ldots, q_n \in \mathbb{F}^\times$ s.t. $q_1^{e_1} \cdots q_n^{e_n} \neq 1$ unless $e_1 = \cdots = e_n = 0$.

Compatible rational functions

$\Delta = \{\delta_1, \ldots, \delta_\ell, \sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n\}$.

Definition. A sequence of rational functions

$$u_1, \ldots, u_\ell, v_1, \ldots, v_m, w_1, \ldots, w_n$$

is compatible w.r.t. Δ if

$$v_1 \cdots v_m w_1 \cdots w_n \neq 0,$$

and the following six sets of CC's hold:

Unmixed	Mixed
$\delta_i(u_j) = \delta_j(u_i), \quad 1 \leq i < j \leq \ell$	$\frac{\delta_i(v_j)}{v_j} = \sigma_j(u_i) - u_i, \quad i = 1, \ldots, \ell, \; j = 1, \ldots, m$
$\frac{\sigma_i(v_j)}{v_j} = \frac{\sigma_j(v_i)}{v_i}, \quad 1 \leq i < j \leq m$	$\frac{\delta_i(w_k)}{w_k} = \tau_k(u_i) - u_i, \quad i = 1, \ldots, \ell, \; k = 1, \ldots, n$
$\frac{\tau_i(w_j)}{w_j} = \frac{\tau_j(w_i)}{w_i}, \quad 1 \leq i < j \leq n$	$\frac{\sigma_j(w_k)}{w_k} = \frac{\tau_k(v_j)}{v_j}, \quad j = 1, \ldots, m, \; k = 1, \ldots, n$

A structure theorem

Theorem. A sequence of rational functions $u_1, \ldots, u_\ell, v_1, \ldots, v_m, w_1, \ldots, w_n$ is compatible iff, for all i, j, k with $1 \leq i \leq \ell$, $1 \leq j \leq m$ and $1 \leq k \leq n$,

$$u_i = \frac{\delta_i(f(\mathbf{t}, \mathbf{x}, \mathbf{y}))}{f(\mathbf{t}, \mathbf{x}, \mathbf{y})} + \sum_{j=1}^{m} x_j \, \frac{\delta_i(\alpha_j(\mathbf{t}))}{\alpha_j(\mathbf{t})} + \beta_i(\mathbf{t}),$$

$$v_j = \frac{\sigma_j(f(\mathbf{t}, \mathbf{x}, \mathbf{y}))}{f(\mathbf{t}, \mathbf{x}, \mathbf{y})} \, \alpha_j(\mathbf{t}) \, \lambda_j(\mathbf{x}),$$

$$w_k = \frac{\tau_k(f(\mathbf{t}, \mathbf{x}, \mathbf{y}))}{f(\mathbf{t}, \mathbf{x}, \mathbf{y})} \, \mu_k(\mathbf{y}),$$

where $f \in \mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$, $\alpha_j, \beta_i \in \mathbb{F}(\mathbf{t})$, $\lambda_j \in \mathbb{F}(\mathbf{x})$, $\mu_k \in \mathbb{F}(\mathbf{y})$,

$\beta_1(\mathbf{t}), \ldots, \beta_\ell(\mathbf{t})$ are compatible w.r.t. $\{\delta_1, \ldots, \delta_\ell\}$,

$\lambda_1(\mathbf{x}), \ldots, \lambda_m(\mathbf{x})$ are compatible w.r.t. $\{\sigma_1, \ldots, \sigma_m\}$, and

$\mu_1(\mathbf{y}), \ldots, \mu_n(\mathbf{y})$ are compatible w.r.t. $\{\tau_1, \ldots, \tau_n\}$.

Survey on unmixed cases

Systems	CC's	Structure
$\begin{cases} \delta_1(z) = u_1\, z \\ \quad\vdots \\ \delta_\ell(z) = u_\ell z, \end{cases}$ $u_1, \ldots, u_\ell \in \mathbb{F}(\mathbf{t})$	$\delta_i(u_j) = \delta_j(u_i),$ $1 \leq i < j \leq \ell$	Christoper's theorem: $u_i = \delta_i(g) + \sum_k c_k \frac{\delta_i(g_k)}{g_k},$ where $c_k \in \mathbb{F}$ and $f, g_k \in \mathbb{F}(\mathbf{t})$
$\begin{cases} \sigma_1(z) = v_1\, z \\ \quad\vdots \\ \sigma_m(z) = v_m\, z, \end{cases}$ $v_1, \ldots, v_m \in \mathbb{F}(\mathbf{x})$	$\frac{\sigma_i(v_j)}{v_j} = \frac{\sigma_j(v_i)}{v_i}$ $1 \leq i < j \leq m$	Ore-Sato's theorem: $v_j = \frac{\sigma_j(g)}{g} \prod_{\mathbf{s} \in S} {}_k\prod_0^{s_j} r_{\mathbf{s}}(\mathbf{s} \cdot \mathbf{x} + k)$ where $f \in \mathbb{F}(\mathbf{x})$, $S \subset \mathbb{Z}^m$ and $r_{\mathbf{s}} \in \mathbb{F}(z)$.
$\begin{cases} \tau_1(z) = w_1\, z \\ \quad\vdots \\ \tau_n(z) = w_n\, z, \end{cases}$ $w_1, \ldots, w_n \in \mathbb{F}(\mathbf{y})$	$\frac{\tau_i(w_j)}{w_j} = \frac{\tau_j(w_i)}{w_i}$ $1 \leq i < j \leq n$	A q-analogue by I. Gel'fand, M. Graev, and V. Retakh. when $q_1 = \cdots = q_n$

Hyperexponential-Hypergeometric elements

Definition. A first-order system

$$\{\delta_i(z) = u_i\, z,\ \sigma_j(z) = v_j\, z,\ \tau_k(z) = w_k\, z \mid i = 1, \ldots, \ell,\ j = 1, \ldots, m,\ k = 1, \ldots, n\} \qquad (*)$$

is compatible if $u_1, \ldots, u_\ell, v_1, \ldots, v_m, w_1, \ldots, w_n$ are compatible w.r.t. Δ.

Definition. A nonzero solution of (*) is called a hyperexponential-hypergeometric element, abbr. as H-element.

Multiplicative decompositions

Corollary. An H-element can be written as

$$\text{a rational function} \times (\alpha_1(\mathbf{t})^{x_1} \cdots \alpha_m(\mathbf{t})^{x_m}) \times \mathcal{E}(\mathbf{t})\, \mathcal{G}(\mathbf{x})\, \mathcal{Q}(\mathbf{y}),$$

where

$\mathcal{E}$ is hyperexp. w.r.t. $\{\delta_1, \ldots, \delta_\ell\}$,

$\mathcal{G}$ hypergeom. w.r.t. $\{\sigma_1, \ldots, \sigma_m\}$,

$\mathcal{Q}$ q-hypergeom. w.r.t. $\{\tau_1, \ldots, \tau_n\}$.

Standard representations

Let $\prec$ be a monomial ordering in $\mathbb{F}[\mathbf{t}, \mathbf{x}, \mathbf{y}]$.

Definition. Given a sequence R of compatible rational functions

$$u_1, \dots, u_\ell,\ v_1, \dots, v_m,\ w_1, \dots, w_n$$

in $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$, a sequence S of rational functions:

$$f, \alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_\ell, \lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_n$$

is the standard representation of R if

(i) the members of R and S satisfy the relations in the structure theorem;

(ii) f has no nontrivial factors in $\mathbb{F}[\mathbf{t}] \cup \mathbb{F}[\mathbf{x}] \cup \mathbb{F}[\mathbf{y}]$;

(iii) $f, \alpha_1, \dots, \alpha_m$ are monic w.r.t. $\prec$.

Computing a representation

For brevity, let $n = 0$.

Algorithm. Given compatible rational functions $u_1, \ldots, u_\ell, v_1, \ldots, v_m$, compute $f \in \mathbb{F}(\mathbf{t}, \mathbf{x})$, $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_l \in \mathbb{F}(\mathbf{t})$, and $\lambda_1, \ldots, \lambda_m \in \mathbb{F}(\mathbf{x}) \in \mathbb{F}(\mathbf{x})$ s.t.

$$u_i = \frac{\delta_i(f)}{f} + \sum_{j=1}^{m} x_j \frac{\delta_i(\alpha_j)}{\alpha_j} + \beta_i \quad \text{for } 1 \leq i \leq l,$$

and

$$v_j = \frac{\sigma_j(f)}{f} \alpha_j \lambda_j \quad \text{for } 1 \leq j \leq m.$$

(1) [Determine α_j and λ_j.]

(1.1) For all j, compute f_j, a_j, b_j s.t.

$$v_j = f_j a_j b_j,$$

where $f_j \in \mathbb{F}(\mathbf{t}, \mathbf{x})$ has no factors in $\mathbb{F}(\mathbf{t}) \cup \mathbb{F}(\mathbf{x})$, $a_j \in \mathbb{F}(\mathbf{t})$ is monic, and $b_j \in \mathbb{F}(\mathbf{x})$;

(1.2) Compute $g_j, r_j \in \mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$ s.t.

$$f_j = \frac{\sigma_j(g_j)}{g_j} r_j \quad \text{with } r_j \text{ being } \sigma_j\text{-reduced;}$$

(1.3) Set $\alpha_j = a_j$ and $\lambda_j = r_j b_j$;

(2) [Determine f.] Set f to be a nonzero solution of the system:

$$\left\{ \sigma_j(z) = \frac{\sigma_j(g_j)}{g_j} z \mid j = 1, \ldots, m \right\}.$$

(3) [Determine β_i] For all i, set

$$\beta_i = u_i - \frac{\delta_i(f)}{f} - \sum_{j=1}^{m} x_j \frac{\delta_i(\alpha_j)}{\alpha_j};$$

(4) Return: $f, \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_\ell, \lambda_1, \ldots, \lambda_m$.

Example

Consider an H-element $h(t,x_1,x_2,x_3,x_4)$ with certificates

$$
\begin{aligned}
u &= \tfrac{2x_1+x_2t-x_2}{t^2-1} & & & u &= x_1\tfrac{\delta(\alpha_1)}{\alpha_1}+x_2\tfrac{\delta(\alpha_2)}{\alpha_2}\\
v_1 &= \tfrac{(t-1)(x_2-x_1)(x_2+x_4-x_1)}{(x_3+x_1+1)(x_1+1)(t+1)} & & & v_1 &= \alpha_1\,\lambda_1\\
v_2 &= \tfrac{(t+1)(x_2+x_4+1)(x_2+x_3+1)}{2(x_2-x_1+1)(x_2+x_4-x_1+1)} & &\rightsquigarrow & v_2 &= \alpha_2\cdot\lambda_2\\
v_3 &= \tfrac{x_2+x_3+1}{x_3+x_1+1} & & & v_3 &= \lambda_3\\
v_4 &= \tfrac{x_2+x_4+1}{x_2+x_4-x_1+1} & & & v_4 &= \lambda_4
\end{aligned}
$$

where

$$\alpha_1=\tfrac{t-1}{t+1},\quad \alpha_2=\tfrac{t+1}{2},$$

$$\lambda_1=\tfrac{(x_2-x_1)(x_2+x_4-x_1)}{(x_3+x_1+1)(x_1+1)},\quad \lambda_2=\tfrac{(x_2+x_4+1)(x_2+x_3+1)}{(x_2-x_1+1)(x_2+x_4-x_1+1)},\quad \lambda_3=\tfrac{x_2+x_3+1}{x_3+x_1+1},\quad \lambda_4=\tfrac{x_2+x_4+1}{x_2+x_4-x_1+1}.$$

A multiplicative form of h is

$$h=\left(\frac{t-1}{t+1}\right)^{x_1}\left(\frac{t+1}{2}\right)^{x_2}\binom{x_2+x_3}{x_2-x_1}\binom{x_2+x_4}{x_1}.$$

Is h in $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$?

Let h be an H-element of the form:

$$h = f(\mathbf{t}, \mathbf{x}, \mathbf{y})\, \alpha_1(\mathbf{t})^{x_1} \cdots \alpha_m(\mathbf{t})^{x_m} \mathcal{E}(\mathbf{t})\, \mathcal{G}(\mathbf{x})\, \mathcal{Q}(\mathbf{y}),$$

where the α_j's are monic

Prop. $h \in \mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y}) \Leftrightarrow \alpha_1 = \cdots = \alpha_m = 1,\ \mathcal{E}(\mathbf{t}) \in \mathbb{F}(\mathbf{t}),\ \mathcal{G}(\mathbf{x}) \in \mathbb{F}(\mathbf{x})$ and $\mathcal{Q}(\mathbf{y}) \in \mathbb{F}(\mathbf{y})$.

Facts:

1. $\mathcal{E}(\mathbf{t}) \in \mathbb{F}(\mathbf{t}) \ \Leftrightarrow\ \frac{\delta_i(\mathcal{E})}{\mathcal{E}} = \frac{\delta_i(g_i)}{g_i}$ for some $g_i \in \mathbb{F}(\mathbf{t})$, $i = 1, \ldots \ell$.
2. $\mathcal{G}(\mathbf{x}) \in \mathbb{F}(\mathbf{x}) \ \Leftrightarrow\ \frac{\sigma_j(\mathcal{G})}{\mathcal{G}} = \frac{\sigma_j(g_j)}{g_j}$ for some $g_j \in \mathbb{F}(\mathbf{x})$, $j = 1, \ldots, m$.
3. $\mathcal{Q}(\mathbf{y}) \in \mathbb{F}(\mathbf{y}) \ \Leftrightarrow\ \frac{\tau_k(\mathcal{Q})}{\mathcal{Q}} = \frac{\tau_k(g_k)}{g_k}$ for some $g_k \in \mathbb{F}(\mathbf{y})$, $k = 1, \ldots, n$.

Is h algebraic over $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$?

Prop. h is algebraic over $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$ iff $h^d \in \mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$ for some $d \in \mathbb{Z}$ with $d \neq 0$.

Corollary. Let h be an H-element of the form:

$$h = f(\mathbf{t}, \mathbf{x})\, \alpha_1(\mathbf{t})^{x_1} \cdots \alpha_m(\mathbf{t})^{x_m} \mathcal{E}(\mathbf{t})\, \mathcal{G}(\mathbf{x}),$$

where the α_j's are monic. Let $\mathbb{U}$ denote the set of roots of unity in $\mathbb{F}$.
Then h is algebraic over $\mathbb{F}(\mathbf{t}, \mathbf{x})$ iff

(i) $\alpha_1 = \cdots = \alpha_m = 1$,

(ii) $\frac{\delta_i(\mathcal{E})}{\mathcal{E}} = r_i \frac{\delta_i(g_i(\mathbf{t}))}{g_i(\mathbf{t})}$, where $r_i \in \mathbb{Q}$, $g_i \in \mathbb{F}(\mathbf{t})$, $i = 1, \ldots, \ell$;

(iii) $\frac{\sigma_j(\mathcal{G})}{\mathcal{G}} = c_j \frac{\sigma_j(g_j(\mathbf{x}))}{g_j(\mathbf{x})}$, where $c_j \in \mathbb{U}$, $g_j \in \mathbb{F}(\mathbf{x})$, $j = 1, \ldots, m$;

Example

Let h satisfy $\delta_1(h) = u_1 h$, $\delta_2(h) = u_2 h$ and $\sigma(h) = vh$, where

$$u_1 = \frac{3t_1+1}{2t_1(t_1+1)}, \quad u_2 = \frac{x-2t_2}{3t_2(x+t_2)}, \quad v = -\frac{x+t_2}{x+t_2+1}$$

- By the structure theorem,

$$f = \frac{1}{x+t_2}, \quad \alpha = 1, \quad \beta_1 = \frac{3t_1+1}{2t_1(t_1+1)}, \quad \beta_2 = \frac{1}{3t_2}, \quad \lambda = -1.$$

- Christopher's theorem

$$\beta_1 = \frac{\delta_1(t_1+1)}{t_1+1} + \frac{1}{2}\frac{\delta_1(t_1)}{t_1}, \quad \beta_2 = \frac{\delta_2(t_1+1)}{t_1+1} + \frac{1}{3}\frac{\delta_2(t_2)}{t_2}.$$

- Observe that $\lambda = -1$.

So h is algebraic over $\mathbb{F}(t_1, t_2, x)$.

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h is of the form

$$h = c\,\frac{1}{x+t_2}\left((t_1+1)t_1^{\frac{1}{2}}\,t_2^{\frac{1}{3}}\right)(-1)^x.$$

Algebraic dependence (ongoing project)

Problem. Given H-elements $h_1, \ldots, h_s$ over $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$, determine whether they are algebraically dependent over $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$.

For $i = 1, \ldots, s$, write

$$h_i = f_i \, \alpha_{i,1}^{x_1} \cdots \alpha_{i,m}^{x_m} \, \mathcal{E}_i(\mathbf{t}) \, \mathcal{G}_i(\mathbf{x}) \, \mathcal{Q}_i(\mathbf{y}),$$

where $\alpha_{i,j} \in \mathbb{F}(\mathbf{t})$ is monic, $\mathcal{E}_i(\mathbf{t})$ hyperexp., $\mathcal{G}_i(\mathbf{x})$ hypergeom., and $\mathcal{Q}_i$ q-hypergeom..

Prop. $h_1, \ldots, h_s$ are a.d. over $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$ iff there exist $e_1, \ldots, e_s \in \mathbb{Z}$, not all zero, s.t.

(i)

$$\alpha_{1,j}^{e_1} \cdots \alpha_{s,j}^{e_s} = 1 \quad \text{for } 1 \le j \le m;$$

(ii)

$$\mathcal{E}_1^{e_1} \cdots \mathcal{E}_s^{e_s} \in \mathbb{F}(\mathbf{t}), \quad \mathcal{G}_1^{e_1} \cdots \mathcal{G}_s^{e_s} \in \mathbb{F}(\mathbf{x}), \quad \mathcal{Q}_1^{e_1} \cdots \mathcal{Q}_s^{e_s} \in \mathbb{F}(\mathbf{y}).$$

Symbolic power functions

Problem. Given $\alpha_{i,j} \in \mathbb{F}(\mathbf{t})$ is monic, where $i = 1, \ldots, s$ and $j = 1, \ldots, m$, compute the $\mathbb{Z}$-submodule $M_{\alpha,j}$ of $\mathbb{Z}^s$ s.t.

$$\alpha_{1,j}^{e_{1,j}} \cdots \alpha_{s,j}^{e_{s,j}} = 1$$

for all $(e_{1,j}, \ldots, e_{s,j}) \in \mathbb{Z}^s$.

A method

1. Determine $M_{\alpha,j}$ —- computing polynomial gcds and solving linear systems over $\mathbb{Z}$.
2. $M_\alpha = M_{\alpha,1} \cap \cdots \cap M_{\alpha,m}$.

Example

Let

$$f_1 = \frac{1}{(t^2-1)(t^3-1)} \quad \text{and} \quad f_2 = \frac{(t-1)^2}{(t+1)^3(t^2+t+1)}.$$

Decide if f_1^x and f_2^x are a.d. over $\mathbb{F}(t,x)$.

By gcd-computation

$$f_1 = \frac{1}{(t-1)^2(t+1)(t^2+t+1)} \quad \text{and} \quad f_2 = \frac{(t-1)^2}{(t+1)^3(t^2+t+1)}.$$

$f_1^{e_1} f_2^{e_2} = 1$ iff

$$\begin{cases} 2e_1 - 2e_2 = 0 \\ e_1 + 3e_2 = 0 \\ e_1 + e_2 = 0 \end{cases}$$

iff $e_1 = e_2 = 0$.

Answer. f_1^x and f_2^x are a.i. over $\mathbb{F}(t,x)$.

Hyperexponential case

Let $\delta = d/dz$

Lemma. Let $\mathcal{E}_1, \ldots, \mathcal{E}_s$ be hyperexponential over $\mathbb{F}(z)$, and

$$\frac{\delta(\mathcal{E}_i)}{\mathcal{E}_i} = \beta_i, \quad i = 1, \ldots, s.$$

Then

$$\exists\, e_1, \ldots, e_s \in \mathbb{Z}, \quad \prod_{i=1}^{s} \mathcal{E}_i^{e_i} \in \mathbb{F}(z)$$

$$\Updownarrow$$

$$\sum_{i=1}^{s} e_i \beta_i \ \text{ is a logarithmic derivative of some element in } \mathbb{F}(z).$$

Example

Let

$$\mathcal{E}_1 = \exp\left(\int \left(\frac{1}{z} - \frac{1}{z-1} + 4z\right) dz\right) \quad \text{and} \quad \mathcal{E}_2 = \exp\left(\int \left(\frac{1}{z} - \frac{2z}{z^2+1} - 2z\right) dt\right).$$

Decide if $\mathcal{E}_1$ and $\mathcal{E}_2$ are a.d. over $\mathbb{F}(z)$.

By partial fraction decomposition,

$$\frac{\delta(\mathcal{E}_1)}{\mathcal{E}_1} = \left(\frac{1}{z} - \frac{1}{z-1}\right) + 4z \quad \text{and} \quad \frac{\delta(\mathcal{E}_2)}{\mathcal{E}_2} = \left(\frac{1}{z} - \frac{2z}{z^2+1}\right) - 2z.$$

$\mathcal{E}^{e_1}\mathcal{E}^{e_2} \in \mathbb{F}(z)$ iff

$$e_1\frac{\delta(\mathcal{E}_1)}{\mathcal{E}_1} + e_2\frac{\delta(\mathcal{E}_2)}{\mathcal{E}_2} \text{ is a logarithemic derivative}$$

iff $4e_1 - 2e_2 = 0$.

Answer. $\mathcal{E}_1$ and $\mathcal{E}_2$ are a.d. over $\mathbb{F}(z)$.

Hypergeometric case

Let $\sigma(f(z)) = f(z+1)$ for all $f \in \mathbb{F}(z)$.

Lemma. Let $\mathcal{G}_1, \ldots, \mathcal{G}_s$ be hypergeometric over $\mathbb{F}(z)$, and

$$\frac{\sigma(\mathcal{G}_i)}{\mathcal{G}_i} = \lambda_i, \quad i = 1, \ldots, s.$$

Write $\lambda_i = c_i \tilde{\lambda}_i$, where c_i is in $\mathbb{F}$ and $\tilde{\lambda}_i$ is monic. Then

$$\exists\, e_1, \ldots, e_s \in \mathbb{Z}, \quad \prod_{i=1}^{s} \mathcal{G}^{e_i} \in \mathbb{F}(z)$$

$$\Updownarrow$$

$$\prod_{i=1}^{s} c_i^{e_i} = 1 \quad \text{and} \quad \sum_{i=1}^{s} e_s \frac{\delta(\lambda_i)}{\lambda_i} = \sigma\left(\frac{\delta(g)}{g}\right) - \frac{\delta(g)}{g}$$

for some $g \in \mathbb{F}(z)$.

Reference. M. Singer. *Deciding if solutions of $\sigma(y_1) = f_1 y_1, \ldots, \sigma y_n = f_n y_n$ are algebraically dependent over $\mathbb{C}(x)$*. A note on discussions at the 2nd NCSU-China Symb. Comput. Collaboration Workshop, Hangzhou, 2007.

Subproblems

1. Given $c_1, \ldots, c_s$ in a finite algebraic extension over $\mathbb{Q}$, find all vectors $(e_1, \ldots, e_s) \in \mathbb{Z}^s$ s.t.

$$\prod_{i=1}^{s} c_i^{e_i} = 1.$$

   Reference. M. Kauers. *Algorithms for Nonlinear Higher Order Difference Equations.* PhD thesis, RISC-Linz, Linz, Austria, 2005.

2. Given $\lambda_1, \ldots, \lambda_s \in \mathbb{F}(z)$, find all vectors $(e_1, \ldots, e_s) \in \mathbb{Z}^s$ s.t.

$$\sum_{i=1}^{s} e_i \frac{\delta(\lambda_i)}{\lambda_i} = \sigma\left(\frac{\delta(g)}{g}\right) - \frac{\delta(g)}{g}$$

   for some $g \in \mathbb{F}(z)$.

   Reference. Singer's note

q-Hypergeometric case

Let $\tau(f(z)) = f(qz)$ for all $f \in \mathbb{F}(z)$, where q is not a root of unity.

Let $\mathcal{Q}_1, \ldots, \mathcal{Q}_s$ be q-hypergeometric over $\mathbb{F}(z)$, and

$$\frac{\tau(\mathcal{Q}_i)}{\mathcal{Q}_i} = \mu_i, \quad i = 1, \ldots, s.$$

Write $\mu_i = c_i \tilde{\mu}_i$, where c_i is in $\mathbb{F}$ and $\tilde{\mu}_i$ is monic.

Lemma. Assume that there exist $e_1, \ldots, e_s \in \mathbb{Z}$ s.t. $\prod_{i=1}^s \mathcal{Q}^{e_i} \in \mathbb{F}(z)$. Then

(i) there exists $g \in \mathbb{F}(z)$ s.t.

$$\sum_{i=1} e_s \frac{\delta(\mu_i)}{\mu_i} = \tau\left(\frac{\delta(g)}{g}\right) - \frac{\delta(g)}{g};$$

(ii)

$$\prod_{i=1}^s c_i^{e_i} = q^d,$$

where $d = \deg \operatorname{num}(g) - \deg \operatorname{den}(g)$.

Summary

$$h = \text{a rational function} \times \alpha_1(\mathbf{t})^{x_1} \cdots \alpha_m(\mathbf{t})^{x_m}\, \mathcal{E}(\mathbf{t})\, \mathcal{G}(\mathbf{x})\, \mathcal{Q}(\mathbf{y}).$$

Future work

1. Implementation of algorithms for decomposing H-elements;
2. An algorithm for determining algebraic dependence of H-elements;
3. Criteria for the existence of telescopers.
4. ...