

Lattice walks in a Weyl chamber of type B

Thomas Feierl

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Outline

Introduction

Proof of the asymptotics for $P_n^+(\mathbf{u} \rightarrow \mathbf{v})$

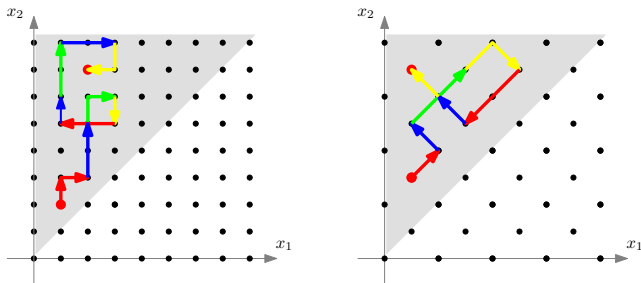
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The model

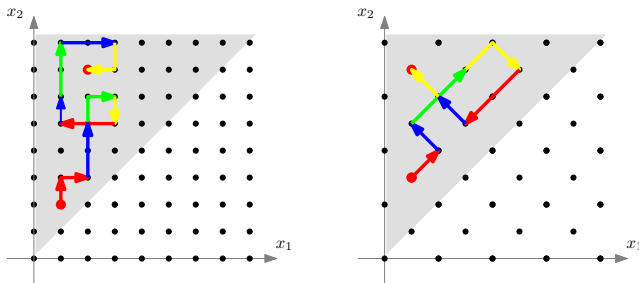


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$$\mathcal{W}^0 = \{(x_1, \dots, x_k) \in \mathcal{L} : 0 < x_1 < \dots < x_k\}.$$

The walks are required to be **reflectable**. (This restricts \mathcal{L} as well as the steps the walks may consist of.)

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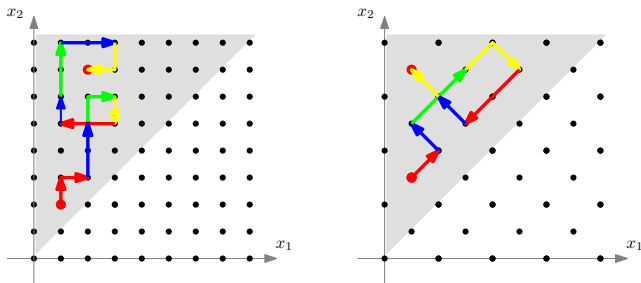


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Motivation for studying these objects:

- ▶ k -noncrossing tangled diagrams
- ▶ Vicious walkers models:
 - ▶ lock step model
 - ▶ random turns model
- ▶ Young tableaux, rhombus tilings, plane partitions, random matrix theory, . . .

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Some notation

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$$\mathcal{W} = \{(x_1, \dots, x_k) \in \mathbb{R}^k : 0 \leq x_1 \leq \dots \leq x_k\}$$

Let $\{\mathbf{b}^{(1)}, \dots, \mathbf{b}^{(k)}\}$ denote the canonical basis in \mathbb{R}^k , and set

$$\Delta = \{\mathbf{b}^{(j+1)} - \mathbf{b}^{(j)} : 1 \leq j < k\} \cup \{\mathbf{b}^{(1)}\}.$$

The set Δ is a *root system* of the reflection group of type B_k generated by the reflections in the hyperplanes

$$x_{j+1} - x_j = 0 \quad \text{for } 1 \leq j < k \quad \text{and} \quad x_1 = 0.$$

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Atomic step sets and composite step sets

Definition

Let $\mathcal{A} \subseteq \mathbb{R}^k$ be a finite set and denote by \mathcal{L} the \mathbb{Z} -lattice spanned by \mathcal{A} . Then the set \mathcal{A} is said to be an **atomic step set** if and only if

- ▶ If $\mathbf{a} \in \mathcal{A}$ then $r_\alpha(\mathbf{a}) \in \mathcal{A}$ for all $\alpha \in \Delta$.
- ▶ If $\mathbf{u} \in \mathcal{W}^0 \cap \mathcal{L}$ and $\mathbf{a} \in \mathcal{A}$ then $\mathbf{u} + \mathbf{a} \in \mathcal{W}$.

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A finite set \mathcal{S} consisting of finite sequences of elements of an atomic step set is said to be an **composite step set** if and only if

$$(\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(m)}) \in \mathcal{S} \implies (r_\alpha(\mathbf{a}^{(1)}), \dots, r_\alpha(\mathbf{a}^{(j)}), \mathbf{a}^{(j+1)}, \dots, \mathbf{a}^{(m)}) \in \mathcal{S}$$

for all $\alpha \in \Delta$ and $j = 1, \dots, m$.

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Some more notation

We will always use the following notation:

- ▶ The set \mathcal{A} denotes an atomic step set.
- ▶ The \mathbb{Z} -lattice spanned by \mathcal{A} is denoted by \mathcal{L} .
- ▶ By \mathcal{S} we denote a composite step set over \mathcal{A} .
- ▶ $w : \mathcal{S} \rightarrow \mathbb{R}_+$ denotes a weight function such that $w(\mathbf{s}) = w(\hat{\mathbf{s}})$ whenever \mathbf{s} and $\hat{\mathbf{s}}$ have the same length (as sequences over \mathcal{A}).

The step generating function

We associate

$$\mathbf{a} = (a_1, \dots, a_k) \in \mathcal{A} \quad \longleftrightarrow \quad \mathbf{z}^{\mathbf{a}} := \prod_{j=1}^k z_j^{a_j}$$

and

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- ▶ The generating function for all paths of length n starting in $\mathbf{0} \in \mathcal{L}$ (and arbitrary end point) is given by $S(z_1, \dots, z_k)^n$.
- ▶ For $\mathbf{c} = (c_1, \dots, c_k) \in \mathcal{L}$, the coefficient $[z_1^{c_1} \dots z_k^{c_k}] S(z_1, \dots, z_k)^n$ equals the weighted sum of all paths of length n from $\mathbf{0} \in \mathcal{L}$ to $\mathbf{c} \in \mathcal{L}$.
- ▶ The generating function for all paths of length n starting in $\mathbf{a} \in \mathcal{L}$ is given by $z^{\mathbf{a}} S(z_1, \dots, z_k)^n$.

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Statement of the problem

Let $\mathbf{u}, \mathbf{v} \in \mathcal{W}^0 \cap \mathcal{L}$. We are interested in

- ▶ $P_n^+(\mathbf{u} \rightarrow \mathbf{v})$, the generating function of n -step walks from \mathbf{u} to \mathbf{v} confined to \mathcal{W}^0 .
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Known results

Answers to special instances of these questions can be found in the literature:

- ▶ Asymptotic growth order of
 - ▶ k -noncrossing tangled diagrams (Zeilberger et al. [1])
 - ▶ Vicious walkers with free end point (Grabiner [2])
- ▶ Precise asymptotics for special configurations of the lock step model of vicious walkers (Krattenthaler [4] et al., Rubey [5])

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Asymptotics for $P_n^+(\mathbf{u} \rightarrow \mathbf{v})$

Theorem

Let \mathcal{M} we denote the set of maximal points of $(\varphi_1, \dots, \varphi_k) \mapsto |S(e^{i\varphi_1}, \dots, e^{i\varphi_k})|$.
We have the asymptotics

$$P_n^+(\mathbf{u} \rightarrow \mathbf{v}) = |\mathcal{M}| S(1, \dots, 1)^n \left(\frac{2}{\pi}\right)^{k/2} \left(\frac{S(1, \dots, 1)}{nS''(1, \dots, 1)}\right)^{k^2+k/2} \\ \times \frac{\left(\prod_{1 \leq j < m \leq k} (v_m^2 - v_j^2)(u_m^2 - u_j^2)\right) \left(\prod_{j=1}^k v_j u_j\right)}{\left(\prod_{j=1}^k (2j-1)!\right)} \left(1 + O(n^{-1/4})\right)$$

as $n \rightarrow \infty$ in the set $\{n : P_n^+(\mathbf{u} \rightarrow \mathbf{v}) > 0\}$.

Asymptotics for $P_n^+(\mathbf{u})$

Theorem

We have the asymptotics

$$P_n^+(\mathbf{u}) = S(1, \dots, 1)^n \left(\frac{2}{\pi}\right)^{k/2} \left(\frac{S(1, \dots, 1)}{nS''(1, \dots, 1)}\right)^{k^2/2} \\ \times \left(\prod_{j=1}^k \frac{u_j(j-1)!}{(2j-1)!}\right) \left(\prod_{1 \leq j < m \leq k} (u_m^2 - u_j^2)\right) \left(1 + O(n^{-1/4})\right)$$

as $n \rightarrow \infty$. Here, $S''(1, \dots, 1)$ denotes the second derivative of $S(z_1, \dots, z_k)$ with respect to any of the z_j .

Introduction

Proof of the asymptotics for $P_n^+(\mathbf{u} \rightarrow \mathbf{v})$

Applications

Outline of the proof

Techniques: reflection principle, generating functions, saddlepoint method

- ▶ Integral formula for the number of “good walks” ($P_n^+(\mathbf{u} \rightarrow \mathbf{v})$).
- ▶ What do composite step sets look like? What properties do the associated step generating functions have?
- ▶ Proof of the main theorem.

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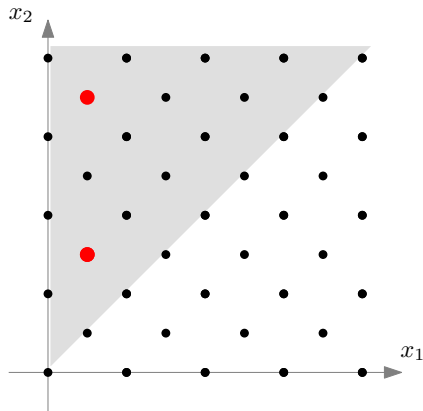
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Exact enumeration: The reflection principle

Theorem (Gessel, Zeilberger)

We have

$$P_n^+(\mathbf{u} \rightarrow \mathbf{v}) = \sum_{r \in B_k} (-1)^{l(r)} P_n(r(\mathbf{u}) \rightarrow \mathbf{v}).$$

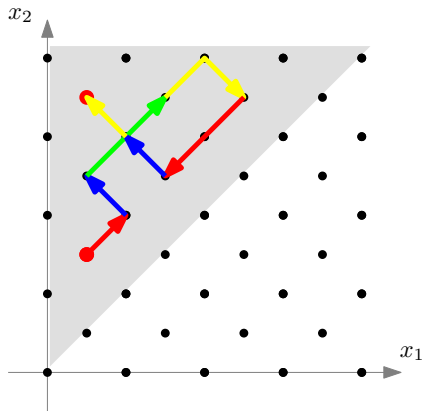


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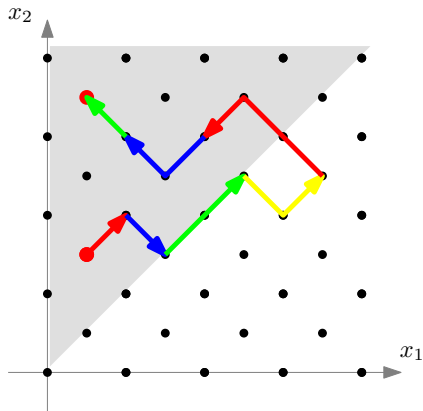


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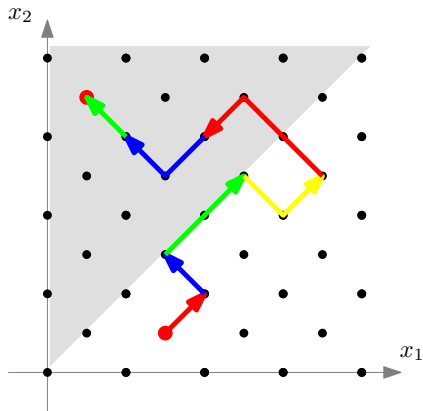


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An integral representation

Lemma

For any two lattice points $\mathbf{u}, \mathbf{v} \in \mathcal{W}^0 \cap \mathcal{L}$ we have

$$P_n^+(\mathbf{u} \rightarrow \mathbf{v}) = \frac{1}{(2\pi i)^k} \int \cdots \int_{|z_1|=\cdots=|z_k|=1} \det_{1 \leq j, m \leq k} \left(z_j^{u_m} - z_j^{-u_m} \right) S(z_1, \dots, z_k)^n \left(\prod_{j=1}^k \frac{dz_j}{z_j^{v_j+1}} \right).$$

Proof of the integral representation

The reflection principle gives us

$$P_n^+(\mathbf{u} \rightarrow \mathbf{v}) = \sum_{\substack{\sigma \in \mathfrak{S}_k \\ \varepsilon_1, \dots, \varepsilon_k \in \{-1, +1\}}} \left(\prod_{j=1}^k \varepsilon_j \right) \operatorname{sgn}(\sigma) \left[z_1^{v_1 - \varepsilon_1 u_{\sigma(1)}} \dots z_k^{v_k - \varepsilon_k u_{\sigma(k)}} \right] S(z_1, \dots, z_k)^n.$$

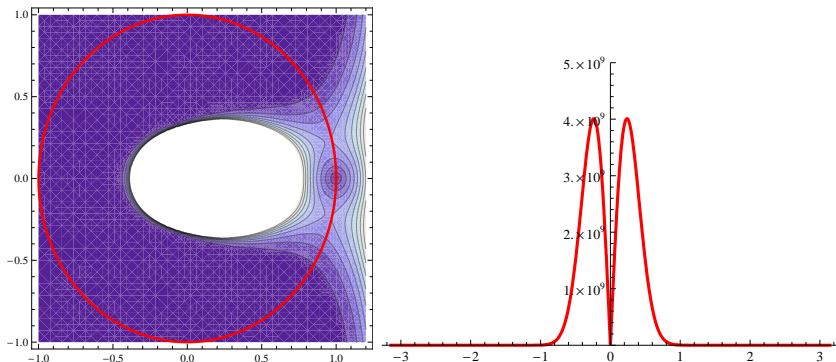
Now, the result follows from Cauchy's integral formula followed by an interchange of summation and integration and the identity

$$\sum_{\substack{\sigma \in \mathfrak{S}_k \\ (\varepsilon_1, \dots, \varepsilon_k) \in \{-1, +1\}^k}} \left(\prod_{j=1}^k \varepsilon_j \right) \operatorname{sgn}(\sigma) \left(\prod_{j=1}^k z_j^{\varepsilon_j u_{\sigma(j)}} \right) = \det_{1 \leq j, m \leq k} \left(z_j^{u_m} - z_j^{-u_m} \right).$$

Example: 2-noncrossing tangled diagrams

For 2-noncrossing tangled diagrams, the integral derived on the previous two pages is given by ($\mathbf{a} = (1, 2)$)

$$P_n^+(\mathbf{a} \rightarrow \mathbf{a}) = \frac{1}{2\pi i} \int_{|z|=1} \left(z - \frac{1}{z}\right) \left(1 + z + \frac{1}{z} + \left(z + \frac{1}{z}\right)^2\right)^n \frac{dz}{z}.$$



Step generating functions

Lemma (Grabiner and Magyar)

If the finite set $\mathcal{G} \subset \mathbb{R}^k$ is reflectable, then \mathcal{G} is isomorphic either to

$$\left\{ \pm \mathbf{b}^{(1)}, \pm \mathbf{b}^{(2)}, \dots, \pm \mathbf{b}^{(k)} \right\} \quad \text{or} \quad \left\{ \sum_{j=1}^k \varepsilon_j \mathbf{b}^{(j)} : \varepsilon_1, \dots, \varepsilon_k \in \{-1, +1\} \right\}.$$

Corollary

The generating function for an atomic step set is either equal to

$$\sum_{j=1}^k \left(z_j + \frac{1}{z_j} \right) \quad \text{or} \quad \prod_{j=1}^k \left(z_j + \frac{1}{z_j} \right).$$

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Step generating functions

Lemma

Let \mathcal{S} be an admissible set of steps. Then there exists a polynomial $P(x)$ with non-negative coefficients such that the associated step generating function $S(z_1, \dots, z_k)$ is equal to either

$$P\left(\sum_{j=1}^k \left(z_j + \frac{1}{z_j}\right)\right) \quad \text{or} \quad P\left(\prod_{j=1}^k \left(z_j + \frac{1}{z_j}\right)\right).$$

Proof (Sketch).

- ▶ If \mathcal{S} contains one step of length ℓ , then \mathcal{S} contains all steps of length ℓ .
- ▶ By definition, all these steps have the same weight.



Step generating functions

Lemma

Let \mathcal{S} be an admissible set of steps. Then there exists a polynomial $P(x)$ with non-negative coefficients such that the associated step generating function $S(z_1, \dots, z_k)$ is equal to either

$$P\left(\sum_{j=1}^k \left(z_j + \frac{1}{z_j}\right)\right) \quad \text{or} \quad P\left(\prod_{j=1}^k \left(z_j + \frac{1}{z_j}\right)\right).$$

Proof (Sketch).

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Step generating functions

Lemma

Let S be an admissible set of steps, and let $S(z_1, \dots, z_k)$ denote the corresponding weighted step generating function. Then, all maxima of the function $(\varphi_1, \dots, \varphi_k) \mapsto |S(e^{i\varphi_1}, \dots, e^{i\varphi_k})|$ lie within the set $\{0, \pi\}^k$. The point $(\varphi_1, \dots, \varphi_k) = (0, \dots, 0)$ is always a maximum.

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From the last lemma we deduce that $S(e^{i\varphi_1}, \dots, e^{i\varphi_k})$ is either of the form $P\left(2\sum_{j=1}^k \cos(\varphi_j)\right)$ or $P\left(2^k \prod_{j=1}^k \cos(\varphi_j)\right)$ for some non-negative polynomial $P(x)$. Consequently, we have $|S(e^{i\varphi_1}, \dots, e^{i\varphi_k})| \leq S(1, \dots, 1)$. □

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Proof of the main theorem

$$\begin{aligned}
 P_n^+(\mathbf{u} \rightarrow \mathbf{v}) &= \frac{1}{(2\pi i)^k} \int_{|z_1|=\dots=|z_k|=1} \det_{1 \leq j, m \leq k} \left(z_j^{u_m} - z_j^{-u_m} \right) S(z_1, \dots, z_k)^n \prod_{j=1}^k \frac{dz_j}{z_j^{v_j+1}} \\
 &= \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \det_{1 \leq j, m \leq k} \left(\sin(u_m \varphi_j) \right) S(e^{i\varphi_1}, \dots, e^{i\varphi_k})^n \left(\prod_{j=1}^k e^{-iv_j \varphi_j} \frac{d\varphi_j}{\pi} \right)
 \end{aligned}$$

The dominant contribution is captured by

$$\sum_{\hat{\varphi} \in \mathcal{M}} \int_{U_n(\hat{\varphi})} \dots \int_{U_n(\hat{\varphi})} \det_{1 \leq j, m \leq k} \left(\sin(u_m \varphi_j) \right) S(e^{i\varphi_1}, \dots, e^{i\varphi_k})^n \left(\prod_{j=1}^k e^{-iv_j \varphi_j} \frac{id\varphi_j}{\pi} \right),$$

where $\mathcal{M} \subseteq \{0, \pi\}^k$ is the set of maxima of $|S(e^{i\varphi_1}, \dots, e^{i\varphi_k})|$, and $U_n(\hat{\varphi}) = \{\varphi \in [0, 2\pi)^k : |\hat{\varphi} - \varphi|_{\infty} < n^{-5/12}\}$.

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Proof of the main theorem

Fix a point $\hat{\varphi} \in \mathcal{M}$ and consider the integral

$$\begin{aligned} & \int_{U_n(\hat{\varphi})} \det_{1 \leq j, m \leq k} \left(\sin(u_m \varphi_j) \right) S(e^{i\varphi_1}, \dots, e^{i\varphi_k})^n \left(\prod_{j=1}^k e^{-iv_j \varphi_j} \frac{id\varphi_j}{\pi} \right) \\ &= \left(\frac{2}{\pi} \right)^k \int_0^{n^{-\frac{5}{12}}} \cdots \int_0^{n^{-\frac{5}{12}}} \det_{1 \leq j, m \leq k} \left((-1)^{(v_m + u_j) \frac{\hat{\varphi}_j}{\pi}} \sin(v_m \varphi_j) \sin(u_j \varphi_j) \right) \\ & \quad \times S(e^{i(\varphi_1 + \hat{\varphi}_1)}, \dots, e^{i(\varphi_k + \hat{\varphi}_k)})^n \left(\prod_{j=1}^k d\varphi_j \right). \end{aligned}$$

Proof of the main theorem

For this range of integration, the Taylor series expansion of $\log |S(e^{i(\varphi_1 + \hat{\varphi}_1)}, \dots, e^{i(\varphi_k + \hat{\varphi}_k)})|$ centered at $\hat{\varphi}$ satisfies

$$\log |S(e^{i(\varphi_1 + \hat{\varphi}_1)}, \dots, e^{i(\varphi_k + \hat{\varphi}_k)})| = \log S(1, \dots, 1) - \Lambda \left(\sum_{j=1}^k \frac{\varphi_j^2}{2} \right) + O(n^{-5/4})$$

as $n \rightarrow \infty$, where $\Lambda = \frac{S''(1, \dots, 1)}{S(1, \dots, 1)} > 0$ and $S''(z_1, \dots, z_k) = \frac{\partial^2}{\partial z_1^2} S(z_1, \dots, z_k)$.

Proof of the main theorem

Consequently, the integral can be rewritten as

$$\left(\frac{2}{\pi}\right)^k S(e^{i\hat{\varphi}_1}, \dots, e^{i\hat{\varphi}_k})^n \\ \times \det_{1 \leq j, m \leq k} \left((-1)^{(v_m + u_j) \frac{\hat{\varphi}_j}{\pi}} \int_0^{n^{-5/12}} \sin(v_m \vartheta) \sin(u_j \vartheta) e^{-n\Lambda \frac{\vartheta^2}{2}} d\vartheta \right) \left(1 + O(n^{-\frac{1}{4}})\right)$$

as $n \rightarrow \infty$.

It can be checked that, whenever we have $P_n^+(\mathbf{u} \rightarrow \mathbf{v}) > 0$, then the factors $(-1)^{(v_m + u_j) \frac{\hat{\varphi}_j}{\pi}}$ can be factorised out of the determinant, and cancel the sign of $S(e^{i\hat{\varphi}_1}, \dots, e^{i\hat{\varphi}_k})^n$.

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Proof of the main theorem

As a consequence of the considerations above we know that, if $\mathbf{u}, \mathbf{v} \in \mathcal{L}$ and $n \in \mathbb{N}$ are chosen such that $P_n^+(\mathbf{u} \rightarrow \mathbf{v}) > 0$, then we have

$$P_n^+(\mathbf{u} \rightarrow \mathbf{v}) = |\mathcal{M}| S(1, \dots, 1)^n \times \det_{1 \leq j, m \leq k} \left(\frac{2}{\pi} \int_0^{n^{-\frac{5}{12}}} \sin(v_m \vartheta) \sin(u_j \vartheta) e^{-n\Lambda \frac{\vartheta^2}{2}} d\vartheta \right) \left(1 + O(n^{-\frac{1}{4}}) \right),$$

where $|\mathcal{M}|$ denotes the number of points of maximum modulus of $S(e^{i\varphi_1}, \dots, e^{i\varphi_k})$.

Proof of the main theorem

Now, the change of variables $u = n\Lambda\vartheta^2/2$ gives

$$\begin{aligned} \frac{2}{\pi} \int_0^{n^{-5/12}} \vartheta^{2\ell} e^{-n\Lambda\vartheta^2/2} d\vartheta &= \frac{2}{n\Lambda\pi} \int_0^{\Lambda n^{1/6}/2} \left(\frac{2u}{n\Lambda}\right)^{\ell-1/2} e^{-u} du \\ &\sim \frac{1}{\pi} \left(\frac{2}{n\Lambda}\right)^{\ell+1/2} \Gamma\left(\ell + \frac{1}{2}\right) \end{aligned}$$

as $n \rightarrow \infty$, which shows that

$$\frac{2}{\pi} \int_0^{n^{-5/12}} \sin(v_m\vartheta) \sin(u_j\vartheta) e^{-n\Lambda\frac{\vartheta^2}{2}} d\vartheta$$

can be expanded into an asymptotic series in decreasing powers of n .

Proof of the main theorem

$$\det_{1 \leq j, m \leq k} \left(\frac{2}{\pi} \int_0^{n^{-\frac{5}{12}}} \sin(v_m \vartheta) \sin(u_j \vartheta) e^{-n\Lambda \frac{\vartheta^2}{2}} d\vartheta \right)$$

Interpreting this determinant as a polynomial (a truncated asymptotic series) with respect to the variables $v_1, \dots, v_k, u_1, \dots, u_k$, we see that it is divisible by

$$\left(\prod_{1 \leq j < m \leq k} (v_m^2 - v_j^2)(u_m^2 - u_j^2) \right) \left(\prod_{j=1}^k v_j u_j \right).$$

The coefficient is easily seen to be

$$\left(\prod_{j=1}^k (2j-1)! \right)^{-2} \left(\frac{2}{n\Lambda} \right)^{k^2+k/2} \det_{1 \leq j, m \leq k} \left(\frac{1}{\pi} \Gamma \left(m + j - \frac{1}{2} \right) \right),$$

where

$$\det_{1 \leq j, m \leq k} \left(\frac{1}{\pi} \Gamma \left(i + j - \frac{1}{2} \right) \right) = \pi^{-k/2} 2^{-k^2} \prod_{j=1}^k (2j-1)!.$$

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$$\det_{1 \leq j, m \leq k} \left(\frac{2}{\pi} \int_0^{n^{-\frac{5}{12}}} \sin(v_m \vartheta) \sin(u_j \vartheta) e^{-n\Lambda \frac{\vartheta^2}{2}} d\vartheta \right)$$

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Asymptotics for $P_n^+(\mathbf{u} \rightarrow \mathbf{v})$

Theorem

Let \mathcal{M} we denote the set of maximal points of $(\varphi_1, \dots, \varphi_k) \mapsto |S(e^{i\varphi_1}, \dots, e^{i\varphi_k})|$.
We have the asymptotics

$$P_n^+(\mathbf{u} \rightarrow \mathbf{v}) = |\mathcal{M}| S(1, \dots, 1)^n \left(\frac{2}{\pi}\right)^{k/2} \left(\frac{S(1, \dots, 1)}{nS''(1, \dots, 1)}\right)^{k^2+k/2} \\ \times \frac{\left(\prod_{1 \leq j < m \leq k} (v_m^2 - v_j^2)(u_m^2 - u_j^2)\right) \left(\prod_{j=1}^k v_j u_j\right)}{\left(\prod_{j=1}^k (2j-1)!\right)} \left(1 + O(n^{-1/4})\right)$$

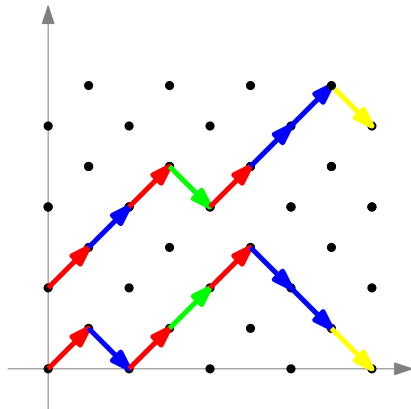
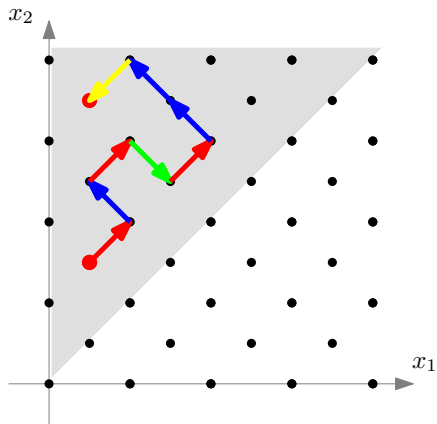
as $n \rightarrow \infty$ in the set $\{n : P_n^+(\mathbf{u} \rightarrow \mathbf{v}) > 0\}$.

Introduction

Proof of the asymptotics for $P_n^+(\mathbf{u} \rightarrow \mathbf{v})$

Applications

Lock step model of vicious walkers: Illustration



Lock step model of vicious walkers: Asymptotics

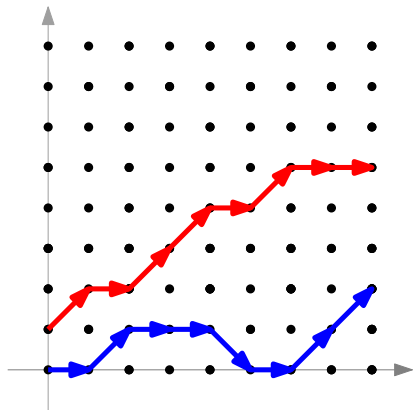
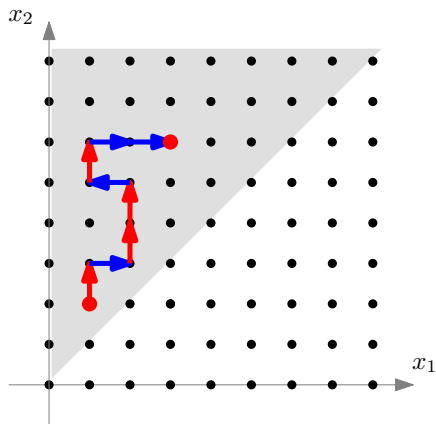
$$S(z_1, \dots, z_k) = \prod_{j=1}^k \left(z_j + \frac{1}{z_j} \right)$$

The number of vicious walkers of length n with k walkers that start at $(0, u_1 - 1), \dots, (0, u_k - 1)$ and end at $(n, v_1 - 1), \dots, (n, v_k - 1)$ (we assume that $u_1 + v_1 \equiv n \pmod{2}$) is asymptotically equal to

$$2^{nk+3k/2} \pi^{-k/2} n^{-k^2-k/2} \frac{\left(\prod_{1 \leq j < m \leq k} (v_m^2 - v_j^2)(u_m^2 - u_j^2) \right) \left(\prod_{j=1}^k v_j u_j \right)}{\left(\prod_{j=1}^k (2j - 1)! \right)}$$

as $n \rightarrow \infty$.

Random turns model of vicious walkers: Illustration



Random turns model of vicious walkers: Asymptotics

$$S(z_1, \dots, z_k) = \sum_{j=1}^k \left(z_j + \frac{1}{z_j} \right)$$

The number of k vicious walkers in the random turns model, where the j -th walker starts at $(0, u_j - 1)$ and, after n steps ends at $(n, v_j - 1)$, is asymptotically equal to

$$2(2k)^n \left(\frac{2}{\pi} \right)^{k/2} \left(\frac{k}{n} \right)^{k^2+k/2} \frac{\left(\prod_{1 \leq j < m \leq k} (v_m^2 - v_j^2)(u_m^2 - u_j^2) \right) \left(\prod_{j=1}^k v_j u_j \right)}{\left(\prod_{j=1}^k (2j - 1)! \right)}$$

as $n \rightarrow \infty$.

$(k + 1)$ -noncrossing tangled diagrams with isolated points

The number of $(k + 1)$ tangled diagrams with isolated points equals the number of simple lattice walks of length n in $0 < x_1 < \dots < x_k$ starting and ending in $\mathbf{a} = (1, 2, \dots, k)$ with step generating function

$$S(z_1, \dots, z_k) = 1 + \left(\sum_{j=1}^k z_j + \frac{1}{z_j} \right) + \left(\sum_{j=1}^k z_j + \frac{1}{z_j} \right)^2.$$

Consequently, we have as $n \rightarrow \infty$

$$P_n^+(\mathbf{a} \rightarrow \mathbf{a}) \sim (1 + 2k + 4k^2)^n \left(\frac{2}{\pi} \right)^{k/2} \left(\frac{1 + 2k + 4k^2}{n(2 + 8k)} \right)^{k^2 + k/2} \left(\prod_{j=1}^k (2j - 1)! \right).$$

$(k + 1)$ -noncrossing tangled diagrams without isolated points

The number of $(k + 1)$ tangled diagrams without isolated points equals the number of simple lattice walks of length n in $0 < x_1 < \dots < x_k$ starting and ending in $\mathbf{a} = (1, 2, \dots, k)$ with step generating function

$$S(z_1, \dots, z_k) = \left(\sum_{j=1}^k z_j + \frac{1}{z_j} \right) + \left(\sum_{j=1}^k z_j + \frac{1}{z_j} \right)^2.$$

Consequently, we have as $n \rightarrow \infty$

$$P_n^+(\mathbf{a} \rightarrow \mathbf{a}) \sim (2k + 4k^2)^n \left(\frac{2}{\pi} \right)^{k/2} \left(\frac{2k + 4k^2}{n(2 + 8k)} \right)^{k^2 + k/2} \left(\prod_{j=1}^k (2j - 1)! \right)$$

as $n \rightarrow \infty$.

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