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The limit shape of large Alternating Sign Matrices

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Six-vertex model, alternating sign matrices, and orthogonal polynomials

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- On the refined 3-enumeration of alternating sign matrices
  `arXiv:math-ph/0404045` *Adv. Appl. Math.* ***34*** *(2005) 798*
- Square ice, alternating sign matrices and classical orthogonal polynomials
  `arXiv:math-ph/0411076` *J. Stat. Mech. 0501 (2005) P005*

Alternating Sign Matrices

$N \times N$ matrix with entries $\in \{0, 1, -1\}$ and such that:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- non-zero entries alternate in sign;
- for each line or column, sum of entries equals 1.

How many of them? $1,\ 2,\ 7,\ 42,\ 429,\ \ldots$

Conjecture: $A_N = \prod_{j=0}^{N-1} \frac{(3j+1)!}{(N+j)!}$ [Mills-Robbins-Rumsey'82]

Proved in '95! [Zeilberger'95]

2 months later, another much simpler proof [Kuperberg'95] which exploits the bijection between ASMs and the configurations of the 6-vertex model with `Domain Wall' b.c.

Thus: $A_N = Z_N$ where Z_N is the partition function of the model with trivial weights

(more on this later)

- Weighted countings: ASMs $q-$ enumeration: $A_N(q)$

  assign weight q^k to an ASM with k "-1" entries

  Explicit answer known for $q=1$ [Zeilberger'95]
  $q=2$ [Mills-Robbins-Rumsey'83]
  $q=3$ [Kuperberg'95]

  The $q=2$ case is closely related to "Domino Tilings of Aztec Diamond" [Jockush-Propp-Shor '98]

  (more on this later)

- Refined countings: ASMs refined $q-$ enumeration: $A_{N,r}(q)$

  count only $N \times N$ ASMs whose sole $+1$ entry in the first row is exactly at the r^{th} position

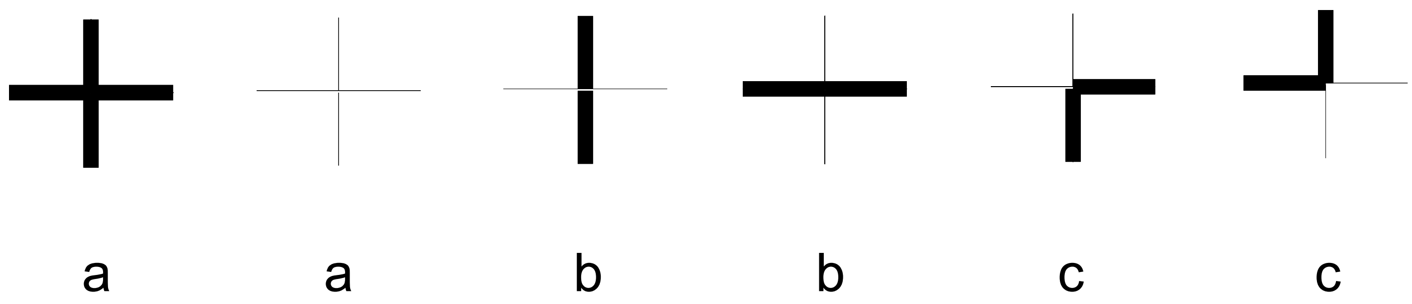
  Explicit answer known only for $q=1$ [Zeilberger'96]
  $q=2$ [Mills-Robbins-Rumsey'83]

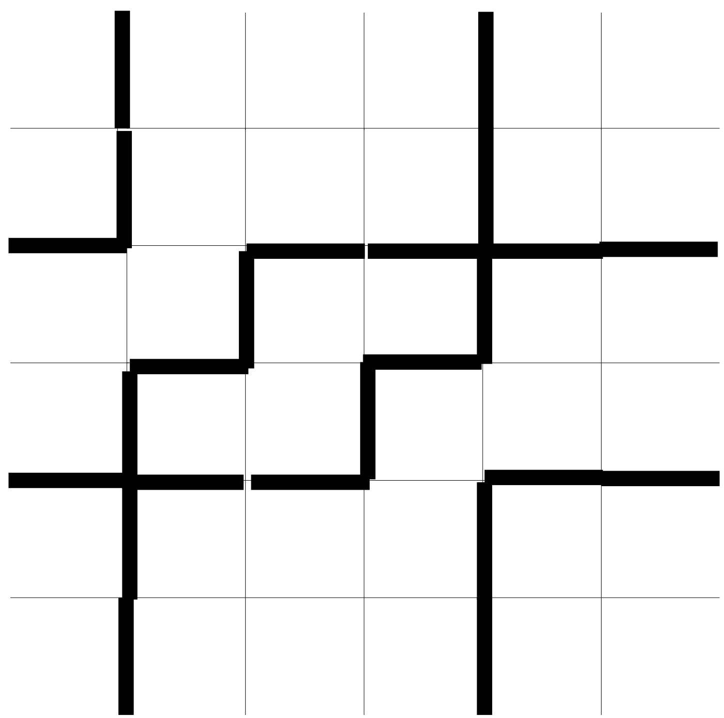
- Doubly refined countings .... $A_{N,r,s}(q)$

	$A_N(q)$	$A_{N,r}(q)$	$A_{N,r,s}(q)$
$q=1$	Zeilberger'95	Zeilberger'96	Stroganov'02
$q=2$	Mills-Robbins-Rumsey'83	M-R-R'83	?
$q=3$	Kuperberg'95	?	?

The six-vertex model

[Lieb '67] [Sutherland'67]



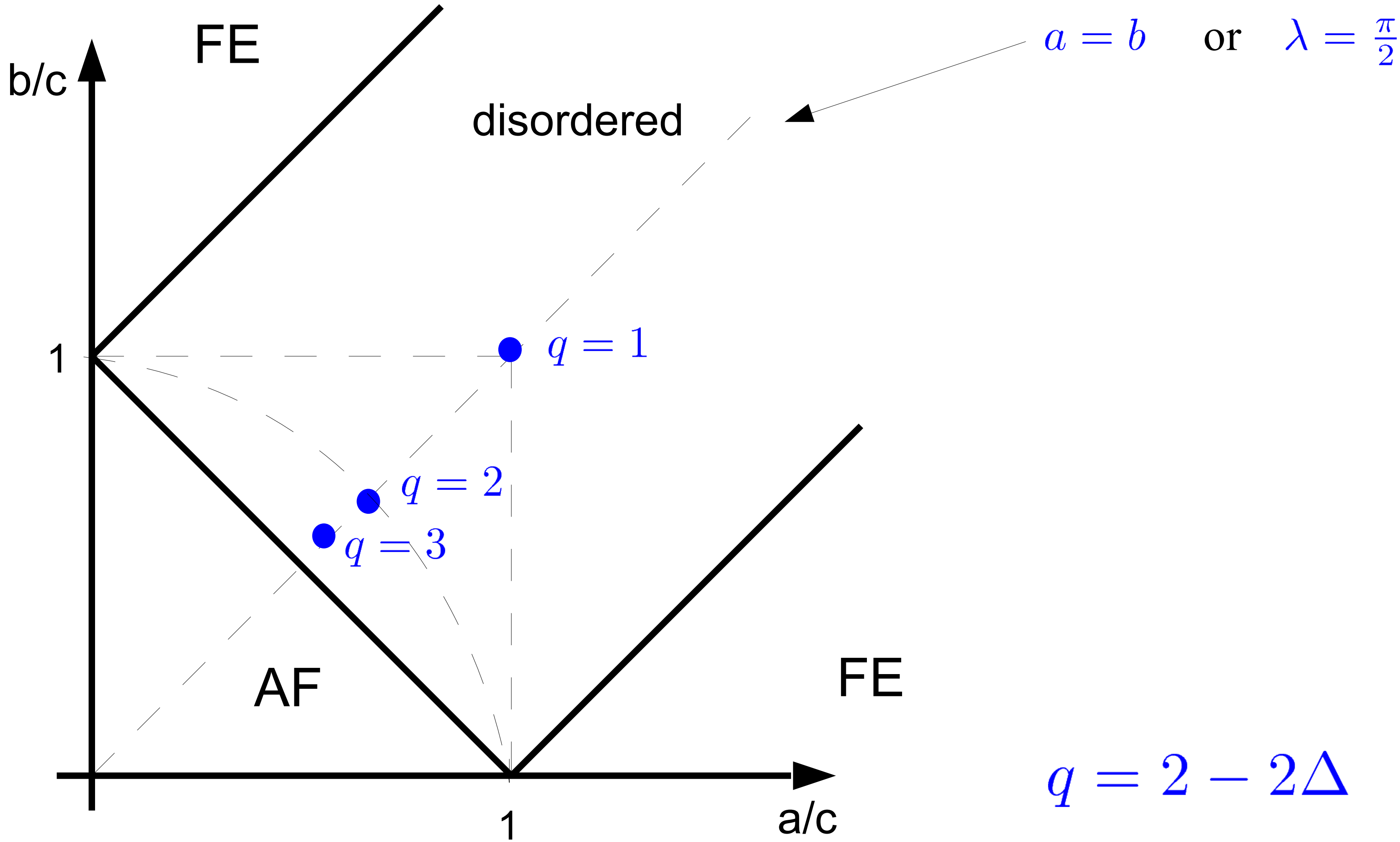


$$
\begin{aligned}
a &= \sin(\lambda + \eta) \\
b &= \sin(\lambda - \eta) \\
c &= \sin 2\eta
\end{aligned}
$$

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}$$

$$t = \frac{b}{a}$$

$Z_N = \sum a^{n_1} b^{n_2} c^{n_3}$ $n_1 + n_2 + n_3 = N^2$ Periodic BC

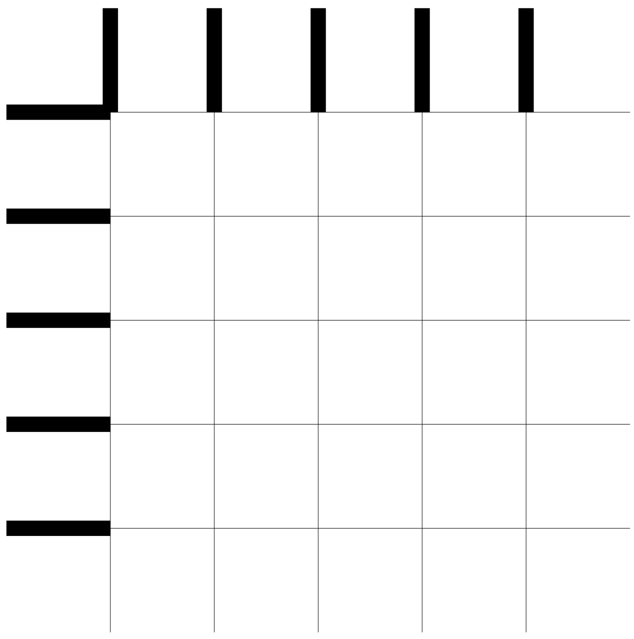


$$q = 2 - 2\Delta$$

The Domain Wall six-vertex model

[Korepin '82]

a	a	b	b	c	c
0	0	0	0	-1	1



The Domain Wall six-vertex model

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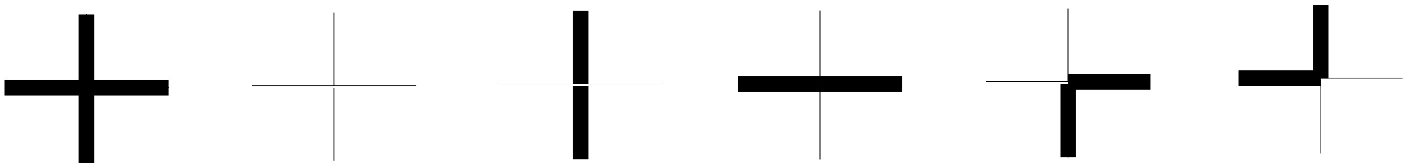

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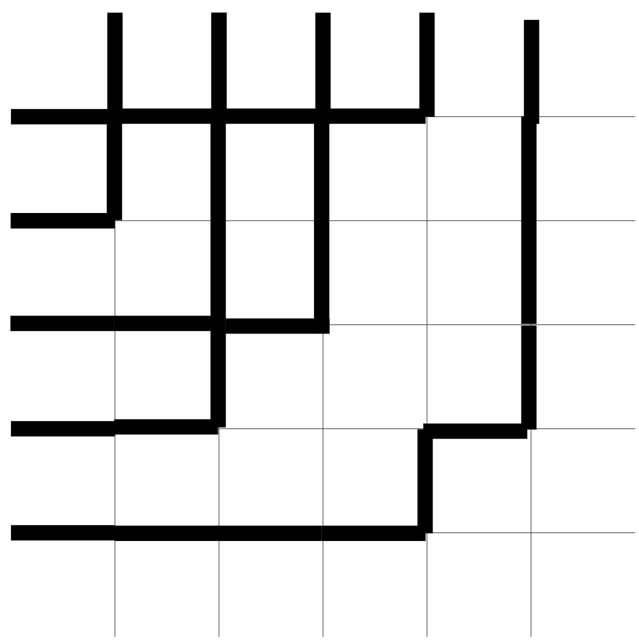


$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

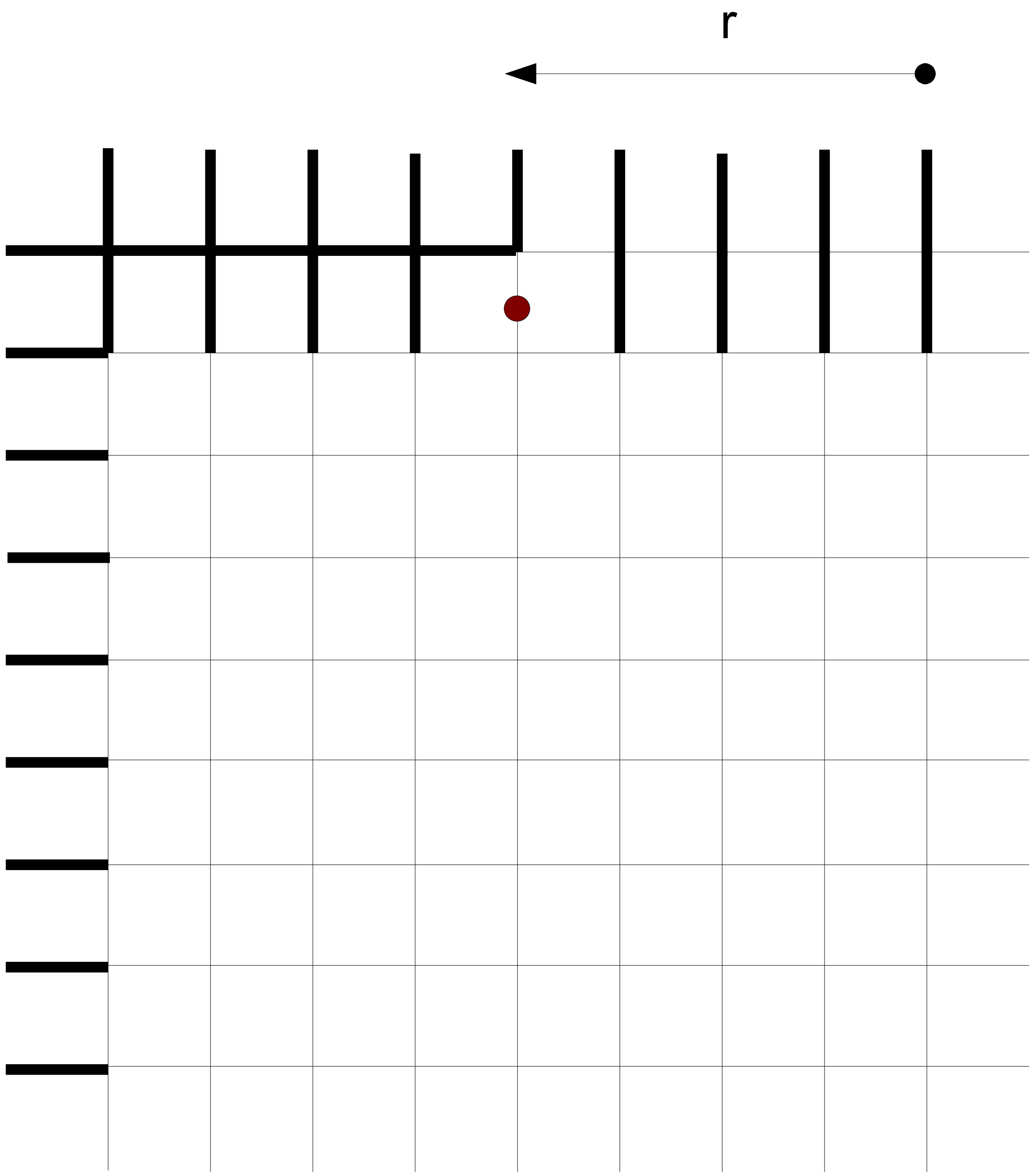
- Partition function [Izergin'87]:

$$Z_N = \frac{[\sin(\lambda+\eta)\sin(\lambda-\eta)]^{N^2}}{\prod_{m=0}^{N-1}(m!)^2} \det_N \left[\partial_\lambda^{j+k-2} \frac{\sin 2\eta}{\sin(\lambda-\eta)\sin(\lambda+\eta)} \right]_{j,k=1}^{N}$$

- One-point boundary correlation function [Bogoliubov-Pronko-Zvonarev'02]:

$$H_N^{(r)}$$

$H_N(r)$



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(analogous expression in terms of Hankel determinants)

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- One-point boundary correlation function [Bogoliubov-Pronko-Zvonarev'02]:

$$H_N^{(r)}$$

(analogous expression in terms of Hankel determinants)

- Two-point boundary correlation function [Colomo-Pronko'05]

(again, expressed in terms of Hankel determinants)

A well-known theorem concerning orthogonal polynomials

Gram determinant: $G_N = \det \left[\int x^{j+k} \mu(x) dx \right]_{j,k=0}^{N-1}$

Let $\{P_j(x)\}_{j=0,1,\ldots}$ a complete set of OP, with respect to the measure $\mu(x)$:

$$\int P_j(x) P_k(x) \mu(x) dx = h_j \delta_{jk}$$

Let k_j be the leading coefficient: $P_j(x) = k_j x^j + \ldots$

Then:

$$\begin{aligned} G_N &= \det \left[\int x^{j+k} \, \mu(x) \, dx \right]_{j,k=0}^{N-1} \\ &= \det \left[\int \frac{P_j(x)}{k_j} \frac{P_k(x)}{k_k} \, \mu(x) \, dx \right]_{j,k=0}^{N-1} \\ &= \prod_{j=0}^{N-1} \frac{h_j}{k_j^2} \end{aligned}$$

Rewriting: $$\varphi(\lambda) := \frac{\sin 2\eta}{\sin(\lambda+\eta)(\lambda-\eta)} = \int_{-\infty}^{+\infty} \mathrm{e}^{(\lambda-\frac{\pi}{2})x} \frac{\sinh \eta x}{\sinh \frac{\pi}{2} x}$$

we have:

$$\partial_\lambda^{j+k} \phi(\lambda)\Big|_{\lambda=\frac{\pi}{2}} = \int_{-\infty}^{+\infty} x^{j+k} \frac{\sinh \eta x}{\sinh \frac{\pi}{2} x} dx$$

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$q = 1 \qquad (\eta = \frac{\pi}{6})$

$$\mu(x) = \frac{1}{4\pi^2} \left| \Gamma\left(\frac{1}{3} + i\frac{x}{6}\right) \Gamma\left(\frac{2}{3} + i\frac{x}{6}\right) \right|^2$$
$$P_n(x) := p_n\left(\tfrac{x}{6}; \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\right) = i^n \left(\tfrac{2}{3}\right)_n \; {}_3F_2\left(\left. \begin{matrix} -n\,,\; n+1\,,\; \frac{1}{3}+i\frac{x}{6} \\ \frac{2}{3}\,,\quad 1 \end{matrix} \right| 1 \right)$$

Continuous Hahn Polynomials !

$$h_n = \frac{2(3n+1)!}{(2n+1)\, 3^{3n+1/2}\, n!}, \qquad \kappa_n = \frac{(2n)!}{6^n\, (n!)^2}$$

$$\Longrightarrow \qquad A_N = \prod_{j=1}^{N-1} \frac{(3j+1)!}{(N+j)!}$$

$q = 2 \qquad (\eta = \frac{\pi}{4})$

$$\mu(x) = \frac{1}{2\pi}\left|\Gamma(\frac{1}{2} + i\frac{x}{4})\right|^2$$

$$P_n(x) := P_n^{(\frac{1}{2})}(\tfrac{x}{4}; \tfrac{\pi}{2}) = i^n \; {}_2F_1\left(\begin{matrix}-n\,,\ \frac{1}{2}+i\frac{x}{4}\\ 1\end{matrix}\middle|\,2\right)$$

Meixner-Pollaczek polynomials

$q = 3 \qquad (\eta = \frac{\pi}{3})$

$$\mu(x) = \frac{1}{8\pi^2}\left|\frac{\Gamma(i\frac{x}{6})\Gamma(\frac{1}{3} + i\frac{x}{6})\Gamma(\frac{2}{3} + i\frac{x}{6})}{\Gamma(\frac{1}{2} + i\frac{x}{6})}\right|^2$$

$$P_n(x) := S_n(\tfrac{x^2}{36}; 0, \tfrac{1}{3}, \tfrac{2}{3}) = \left(\tfrac{1}{3}\right)_n \left(\tfrac{2}{3}\right)_n \; {}_3F_2\left(\begin{matrix}-n\,,\ i\frac{x}{6}\,,\ -i\frac{x}{6}\\ \frac{1}{3}\,,\ \frac{2}{3}\end{matrix}\middle|\,1\right)$$

Continuous Dual Hahn polynomials

For $q = 1, 2, 3$ and only for these values, $q-$enumeration is related to some classical (in the sense of belonging to Askey scheme) orthogonal polynomials

This classical OP structure (in particular, the fact that they obey some known finite difference equation) give rise to recurrence relations for refined $q-$ enumerations, which can be turned into differential eqs. for their generating functions. These can be solved.

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	$A_N(q)$	$A_{N,r}(q)$	$A_{N,r,s}(q)$
$q=1$	Zeilberger'95	Zeilberger'96	Stroganov'02
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INRIA, Paris-Rocquencourt, October 2009

The limit shape of large Alternating Sign Matrices

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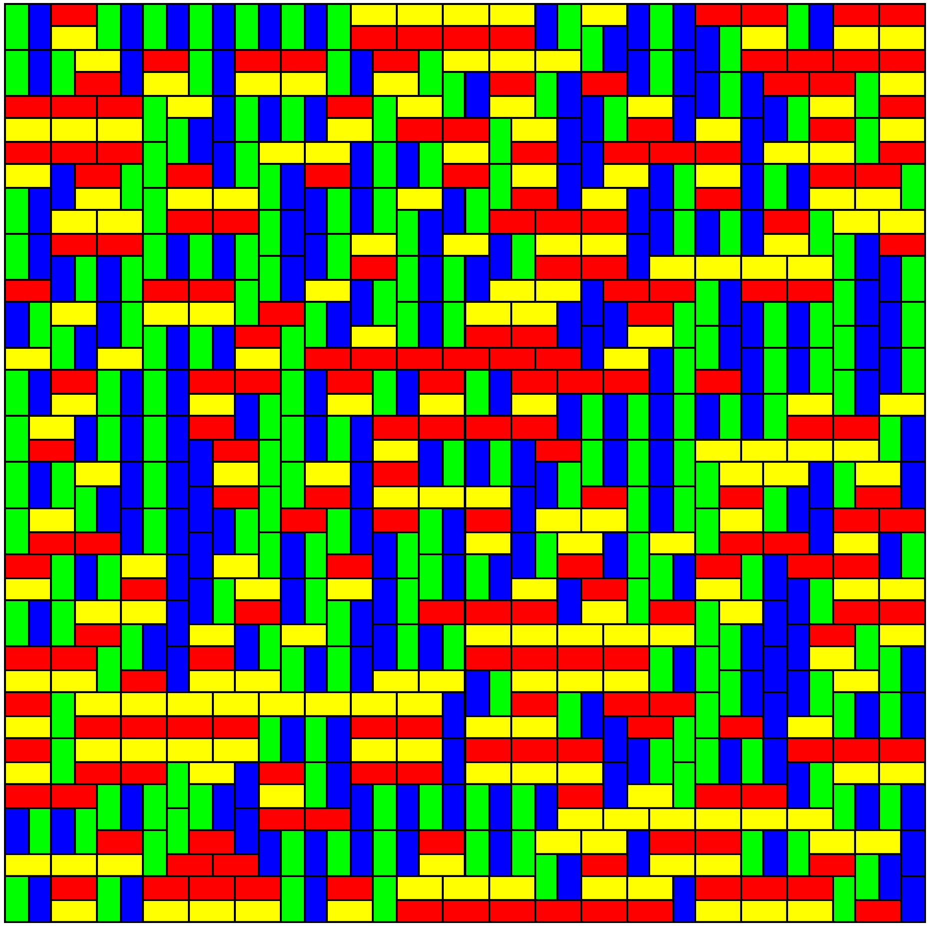
- Emptiness Formation Probability in the domain wall six-vertex model, `arXiv:0712.1524` *Nucl. Phys. B 798 (2008) 340*
- The Arctic Circle revisited, `arXiv:0704.0362` *Contemp. Math. 458 (2008) 361*
- The limit shape of large Alternating Sign Matrices, `arXiv:0803.2697` subm. to *SIAM J. Discr. Math.*
- The Arctic curve of the domain-wall six-vertex model, `arXiv:0907.1264` subm. to *Comm. Math. Phys.*

Domino Tiling of a square:

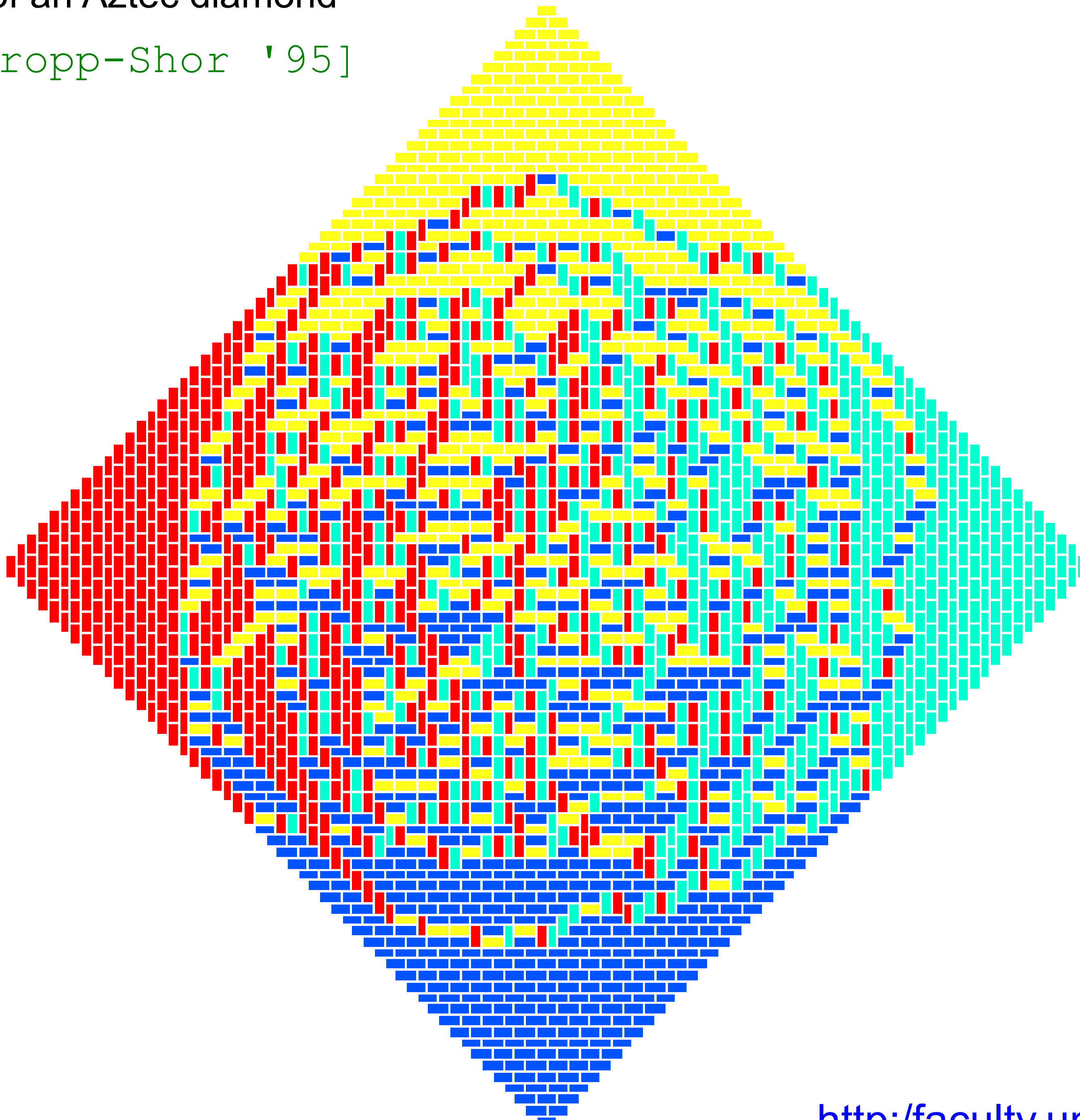


http://www.math.wisc.edu/~propp

Domino tiling of an Aztec diamond

[Jockush-Propp-Shor '95]

$N = 64$



http:/faculty.uml.edu/jpropp

The Arctic Circle Theorem

[Jockush-Propp-Shor '95]

$\forall \varepsilon > 0, \quad \exists N$ such that "almost all" (i.e. with probability $P > 1 - \varepsilon$) randomly picked domino tilings of $AD(N)$ have a temperate region whose boundary stays uniformly within distance εN from the circle of radius $N/\sqrt{2}$.

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Fluctuations:

- boundary fluctuations $N^{1/3}$ [Johansson'00]
- fluctuations of boundary intersection with main diagonal obey Tracy-Widom distribution [Johansson'02]
- after suitable rescaling, boundary has limit as a random function, governed by an Airy stochastic process [Johansson'05]

The Arctic Circle Theorem

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Example of more general phenomena: phase separation, limit shapes, frozen boundaries/arctic curves, e.g:

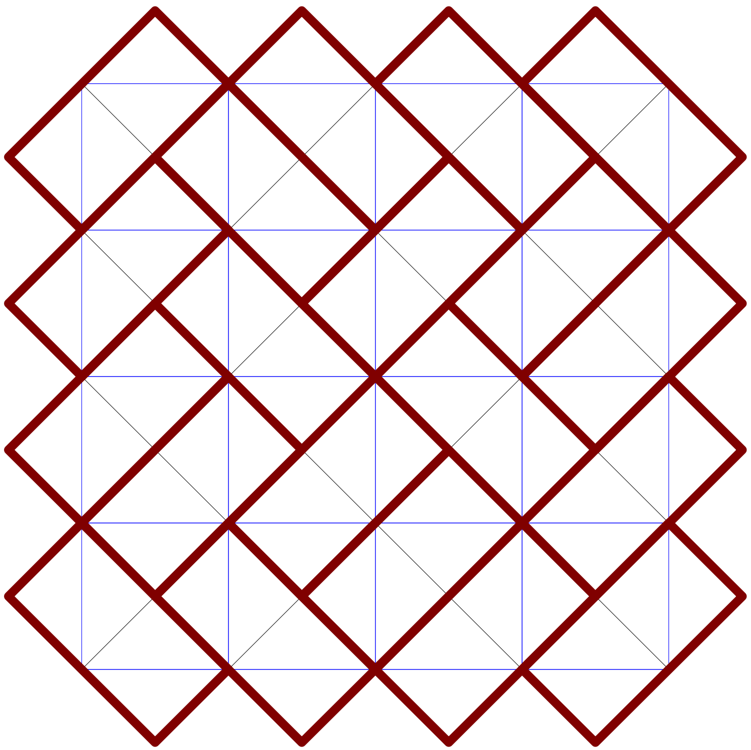
- Young diagrams [Kerov-Vershik '77] [Logan-Shepp '77]
- Boxed plane partitions [Cohn-Larsen-Propp '98]
- Corner melting of a crystal [Ferrari-Spohn '02]
- Plane partitions [Cerf-Kenyon'01][Okounkov-Reshetikhin'01]
- Skewed plane partitions [Okounkov-Reshetikhin '05]

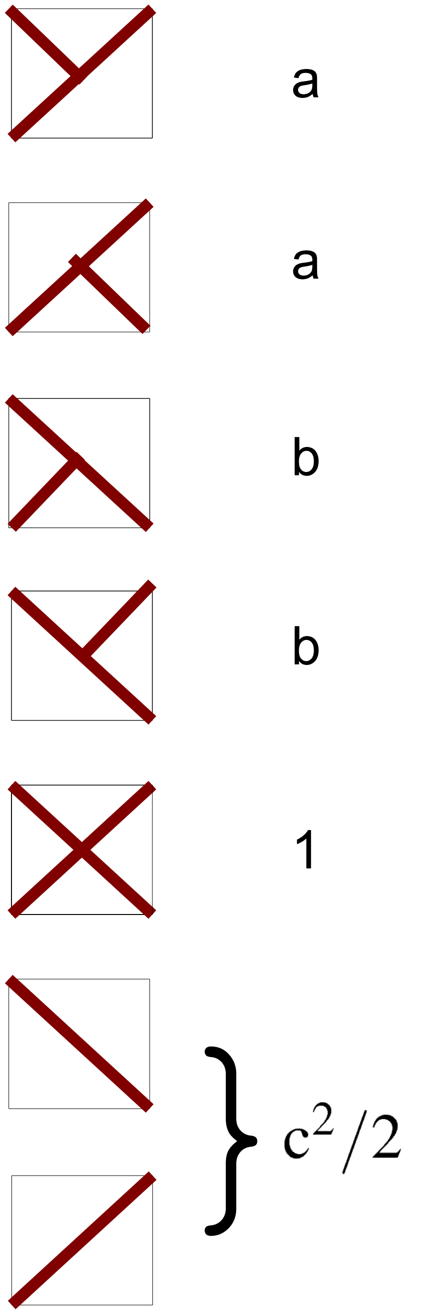
Dimer models and algebraic geometry

[Kenyon, Sheffield, Okounkov, '03-'05]

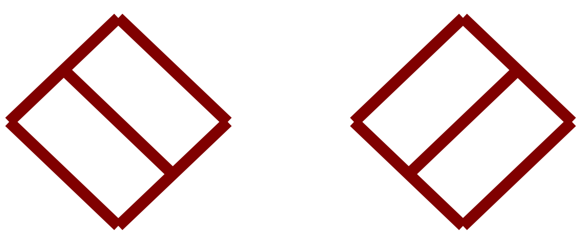
The DW 6VM as a model of interacting dimers

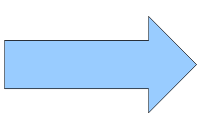
[Elkies-Kuperberg-Larsen-Propp'92]





DW 6VM partition function can be seen as a weighted enumeration of the Domino Tilings of Aztec Diamond; in particular a weight $c^2/2$ is assigned to configurations:

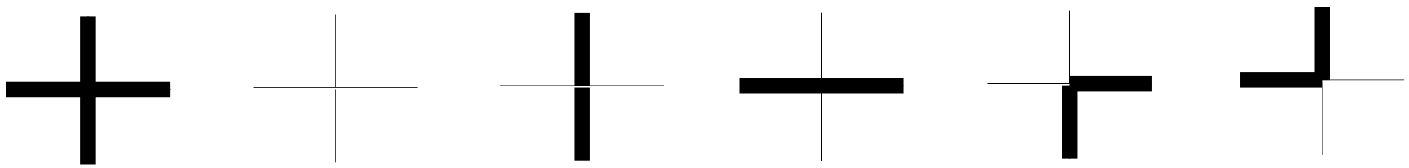




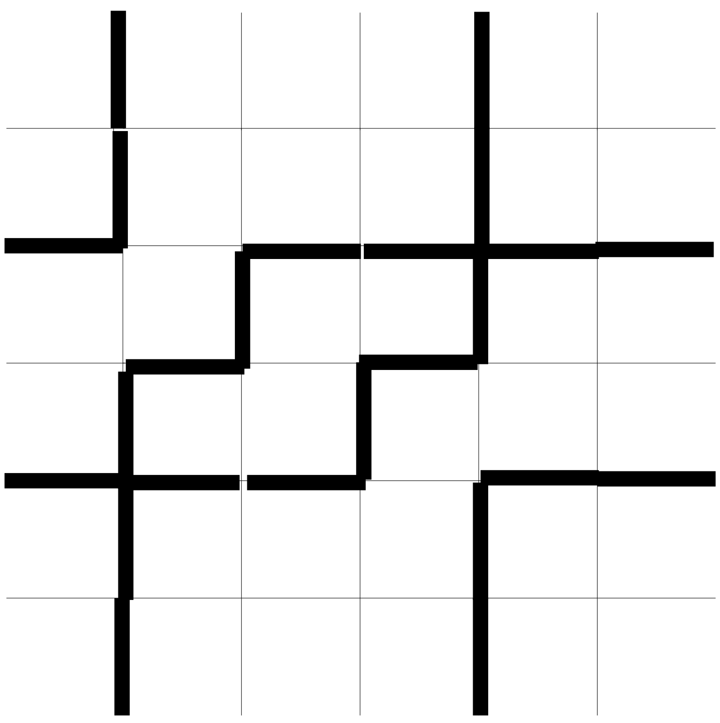
DW 6VM can be seen as a model of interacting dimers on Aztec Diamond.

The six-vertex model

[Lieb '67] [Sutherland'67]



a a b b c c



$$
\begin{aligned}
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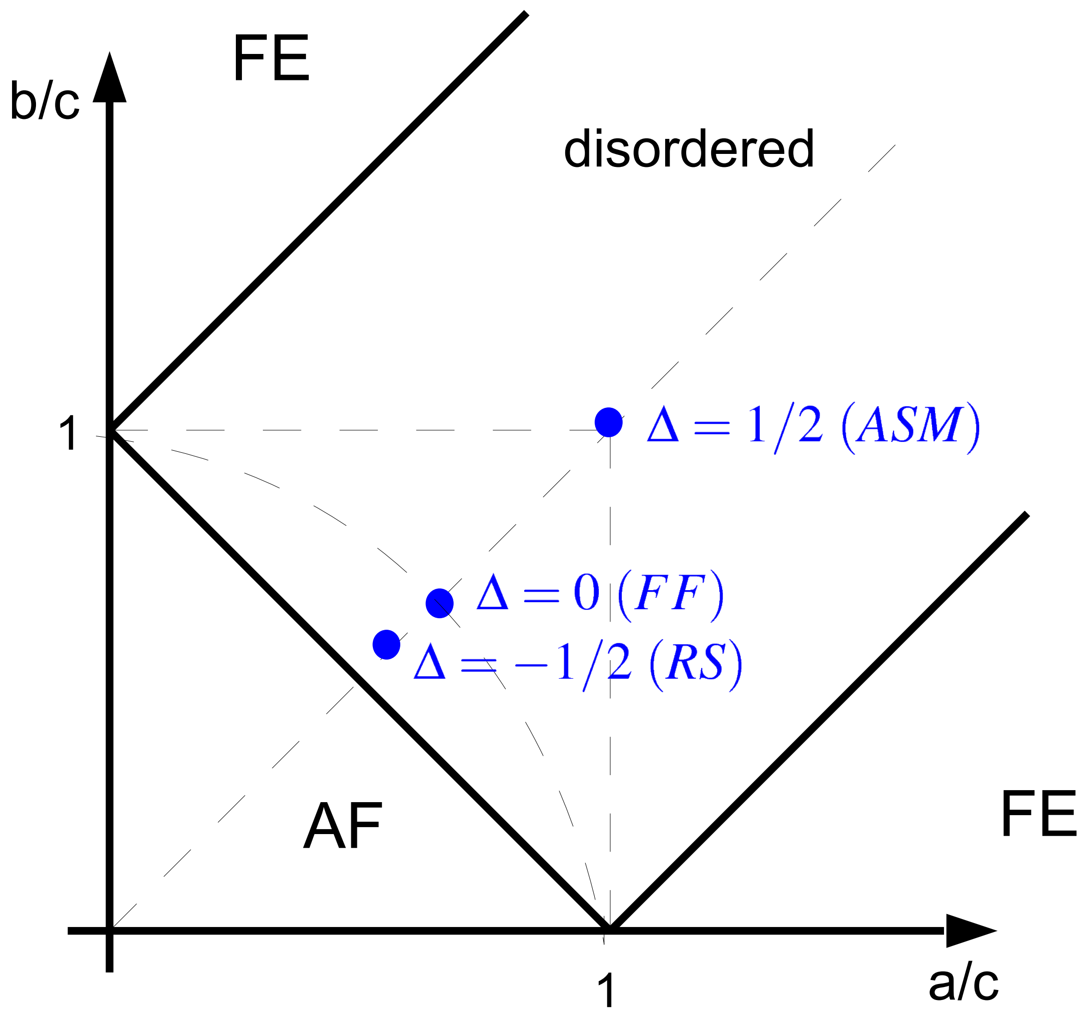
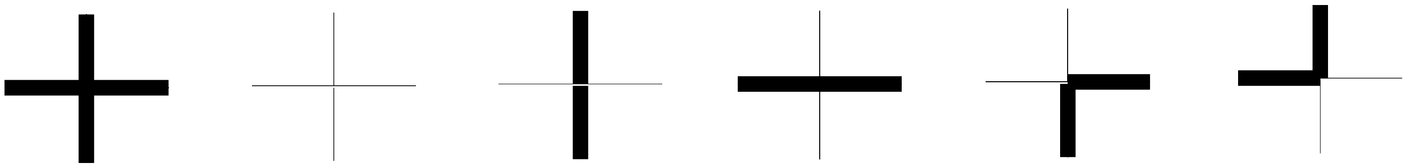
$$t = \frac{b}{a}$$

$Z_N = \sum a^{n_1} b^{n_2} c^{n_3}$ $\qquad$ $n_1 + n_2 + n_3 = N^2$ $\qquad$ Periodic BC

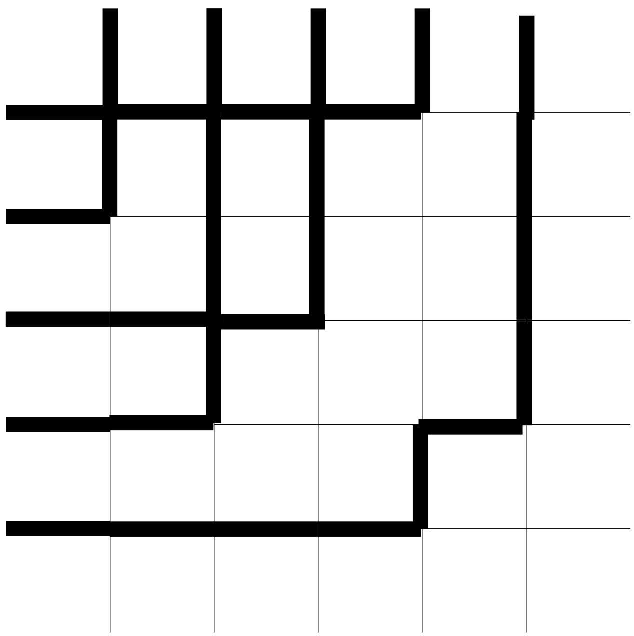


The Domain Wall six-vertex model

[Korepin '82]



a	a	b	b	c	c
0	0	0	0	-1	1



$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Domain Wall six vertex model: known results

- Izergin'87: I-K determinant representation and Hankel determinant representation for Z_N ;
- Bogoliubov-Pronko-Zvonarev'02: one point boundary correlation function;
- Colomo-Pronko'05: two point boundary correlation function.

All above results have rather implicit form, in terms of determinants.

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All above results have rather implicit form, in terms of determinants.

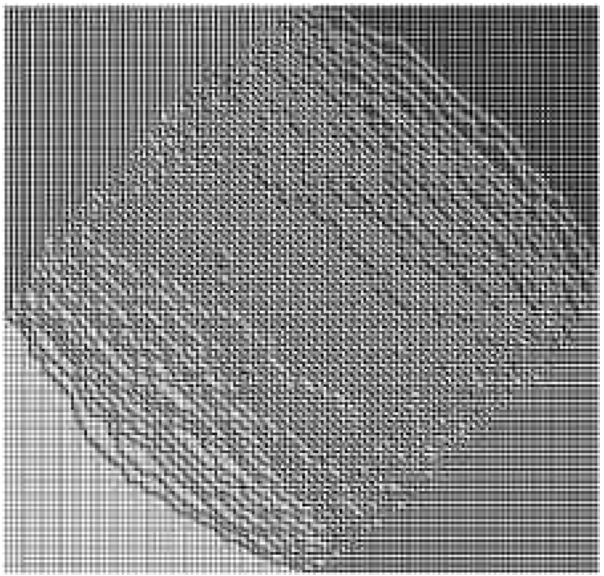
- Korepin Zinn-Justin'00, Zinn-Justin'01, Bleher-Fokin'05, Bleher-Liechty'07-09: Large N behaviour of Z_N:

Bulk free energy: DWBC $\neq$ PBC

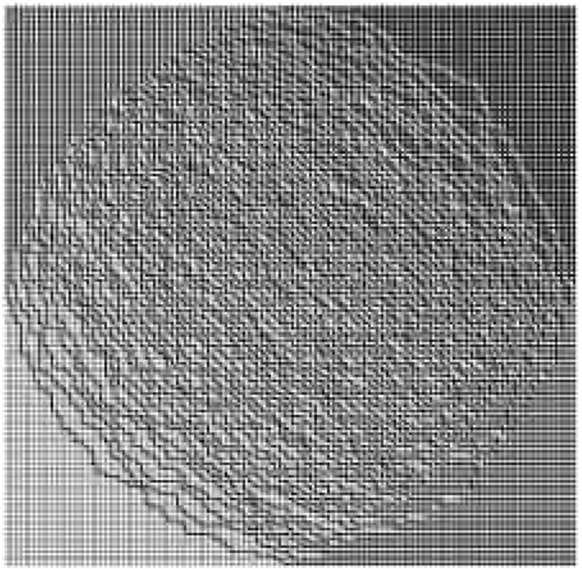
In addition, there are many other results, of more explicit form, for the three specific cases of $\Delta = 0,\ 1/2,\ -1/2$.

Domain Wall six vertex model: numerical results

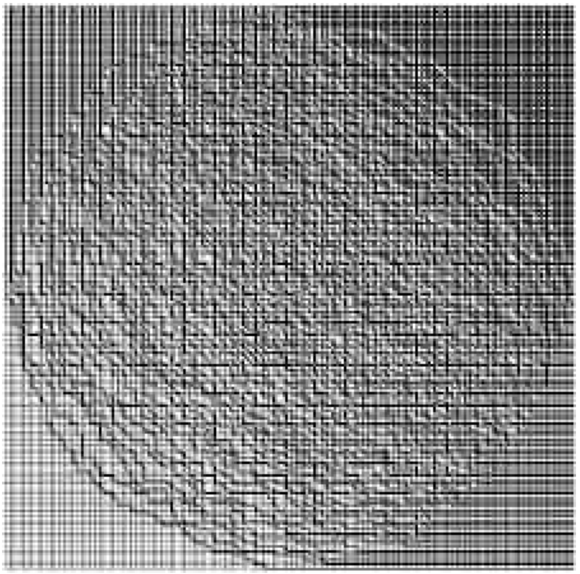
[Eloranta'99] [Zvonarev-Syluasen'04] [Allison-Reshetikhin'05]



$\Delta = -3$



$\Delta = 0$ (free fermions)



$\Delta = 0.707$

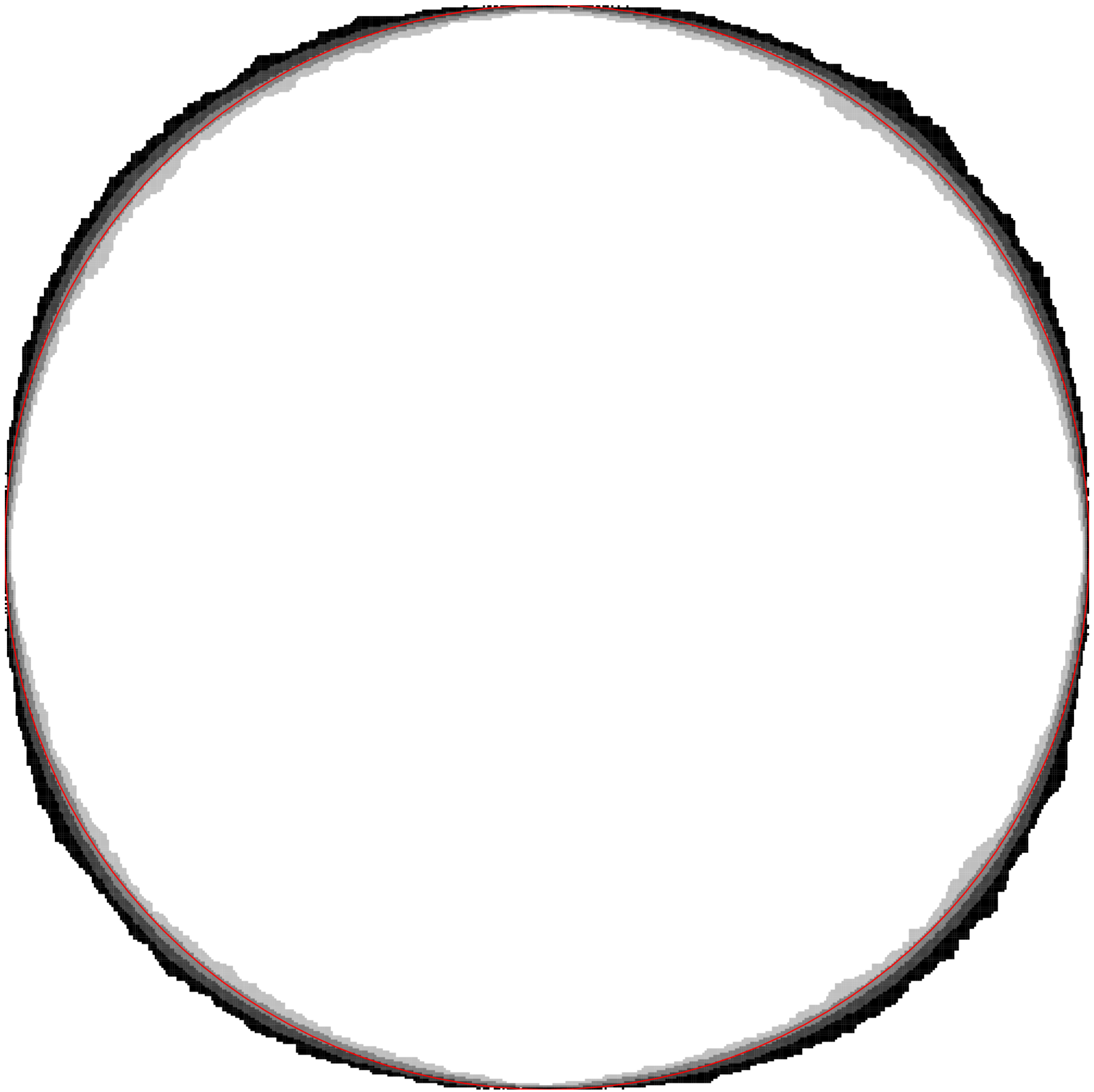
$N = 225$

[Allison-Reshetikhin'05]

ASMs: N=500

199 samples

$\Delta = 1/2$



Ben Wieland (October 2007)

http://www.math.brown.edu/~wieland

The problem

Extend the Arctic Circle Theorem [DWBC 6VM at $\Delta = 0$] to generic values of Δ (including e.g. $\Delta = \frac{1}{2}$: limit shape of ASMs).

- Compute a suitable bulk correlation function

$$F_N(r,s) \qquad\qquad 1 \leq r,s \leq N$$

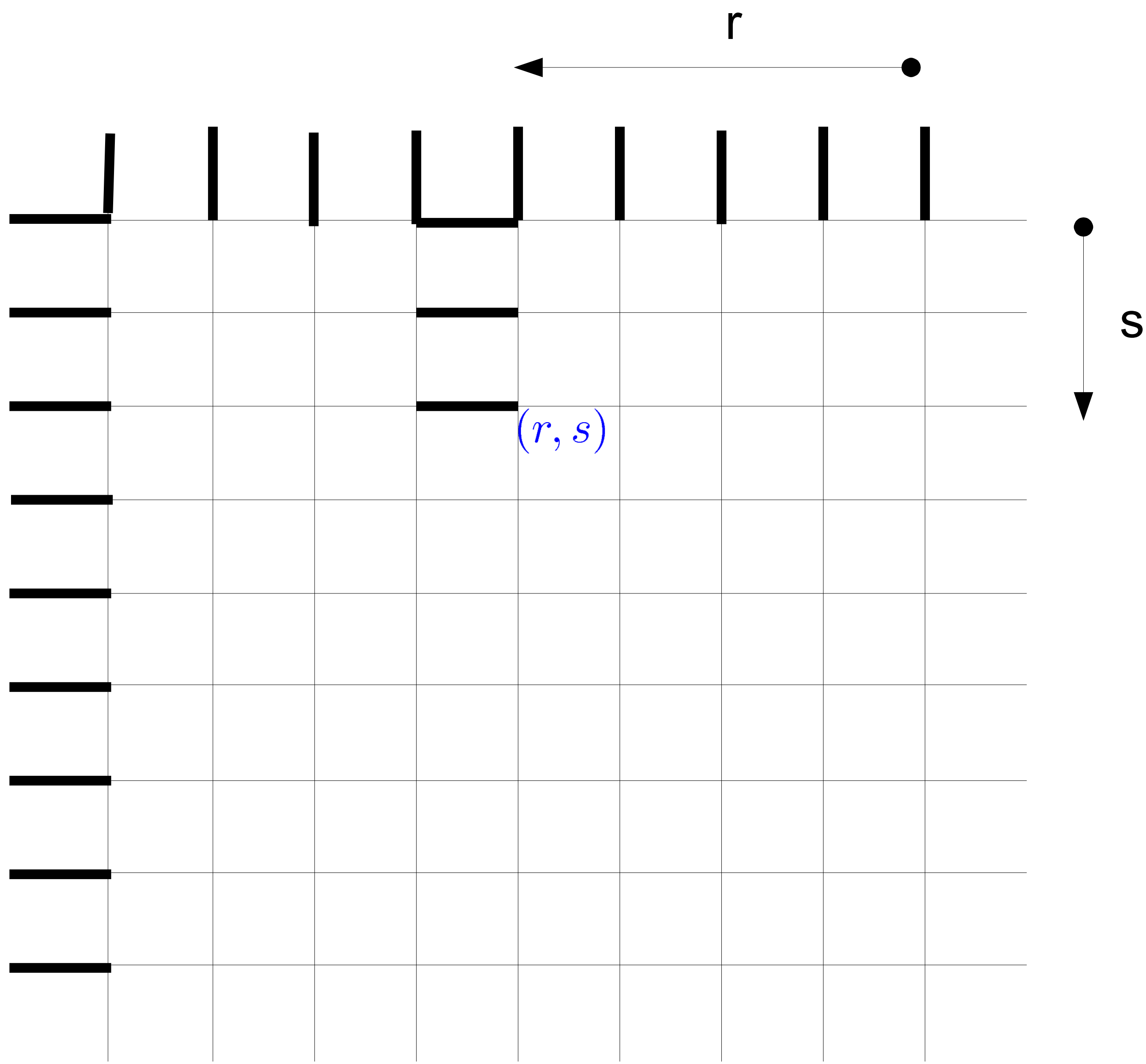
- Evaluate it in the “scaling” limit:

$$N,r,s \longrightarrow \infty \qquad\qquad \frac{r}{N} = x \qquad \frac{s}{N} = y$$

I.e.: evaluate asymptotic behaviour of

$$F(x,y) := \lim_{N\to\infty} F_N(xN,yN) \qquad\qquad x,y \in [0,1]$$

$F_N(r, s)$ Emptiness Formation Probability (EFP)



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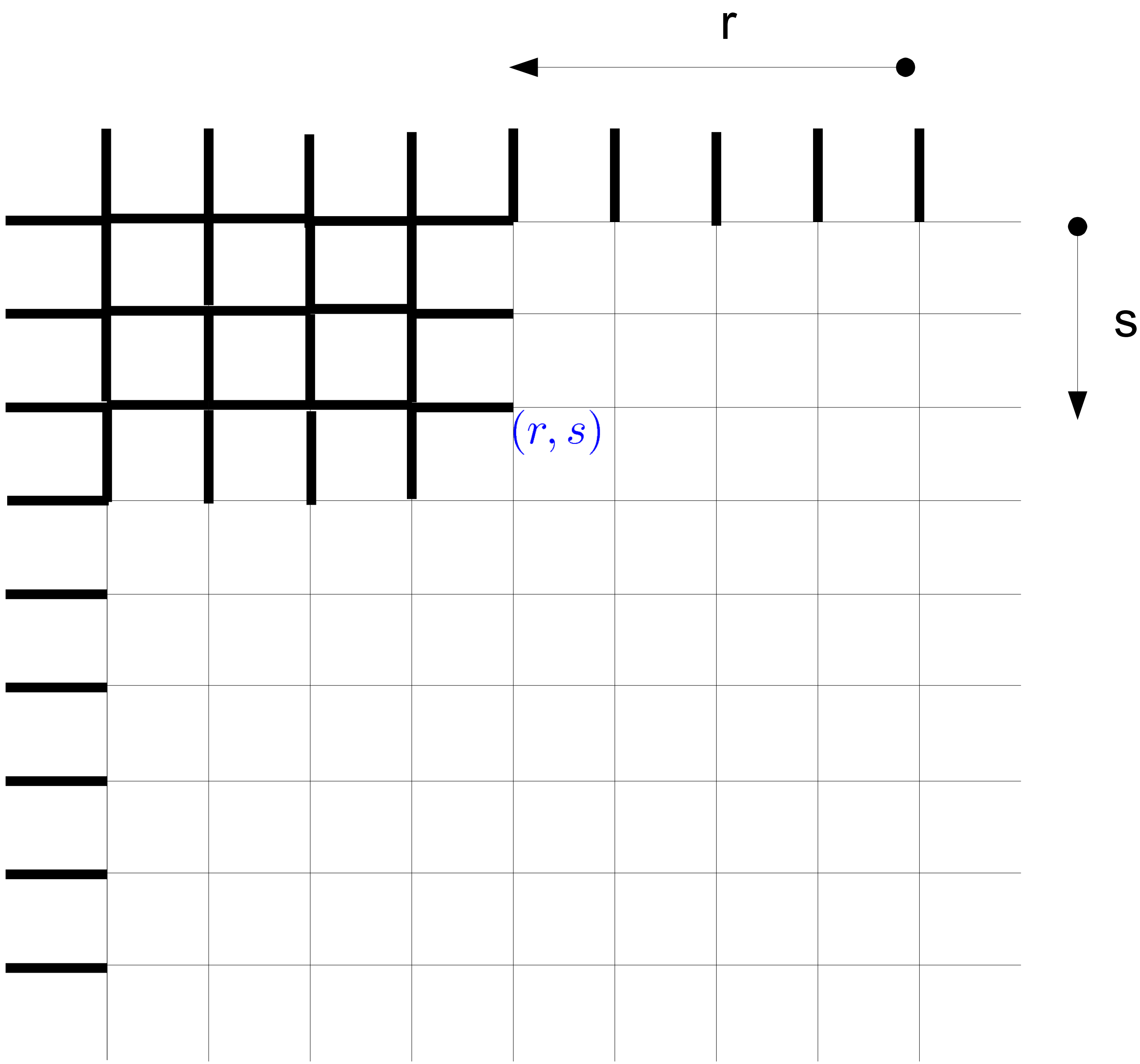

Multiple Integral Representation for EFP

Define the generating function for the 1-point boundary correlator:

$$h_N(z) := \sum_{r=1}^{N} H_N(r) z^{r-1}\,, \qquad h_N(1) = 1\,.$$

Now define, for $s = 1, \dots, N$:

$$h_N^{(s)}(z_1, \dots, z_s) := \frac{1}{\Delta_s(z_1, \dots, z_s)} \det_{1 \le j,k \le s} \left[h_{N-s+k}(z_j)(z_j - 1)^{k-1} z_j^{s-k} \right]$$

- The functions $h_N^{(s)}(z_1, \dots, z_s)$ are totally symmetric polynomials of order $N-1$ in $z_1, \dots, z_s$.
- They encode the full functional dependence of the partially inhomogeneous partition function from its spectral parameters.

Two important properties of $h_N^{(s)}(z_1, \dots, z_s)$:

$$h_N^{(s)}(z_1, \dots, z_{s-1}, 0) = h_N(0) h_{N-1}^{(s-1)}(z_1, \dots, z_{s-1})\,,$$

$$h_N^{(s)}(z_1, \dots, z_{s-1}, 1) = h_N^{(s-1)}(z_1, \dots, z_{s-1})\,.$$

NB: An explicit expression of $h_N(z)$ is known for $\Delta = 0, 1/2, -1/2$.

The following Multiple Integral Representation is valid for EFP ($r, s = 1, 2, \dots, N$):

$$F_N(r,s) = \left(-\frac{1}{2\pi i}\right)^s \oint_{C_0} \dots \oint_{C_0} d^s z \, h_N^{(s)}(z_1, \dots, z_s) \prod_{j=1}^{s} \frac{1}{z_j^r (z_j - 1)^s} \times \prod_{1 \le j < k \le s} \frac{(\tilde{z}_j - 1)(z_k - 1)(z_j - z_k)}{\tilde{z}_j z_k - 1} \, .$$

where

$$\tilde{z}_j := \frac{t^2 z_j}{2\Delta t z_j - 1} \, , \qquad j = 1, \dots, s \, .$$

The contours C_0 are simple anticlockwise contours, enclosing $z = 0$ and no other singularity of the integrand.

Ingredients:

- Quantum Inverse Scattering Method to obtain a determinant representation on the lines of Izergin-Korepin formula;
- Orthogonal Polynomial and Random Matrices technologies to rewrite it as a multiple integral.

Free Fermion point

In this case:

$$\Delta = 0 \qquad\qquad t = 1 \qquad\qquad \tilde{z}_j = -z_j$$

Moreover in this case function $h_N(z)$ is exactly known:

$$h_N(z) = \left(\frac{1+z}{2}\right)^{N-1}$$

MIR for EFP reduces simply to

$$F_N(r,s) = \frac{(-1)^{s(s+1)/2}}{s!2^{s(N-s)}(2\pi i)^s} \oint_{C_0} \cdots \oint_{C_0} d^s z \prod_{1\le j<k\le s} (z_j - z_k)^2 \prod_{j=1}^{s} \frac{(z_j+1)^{N-s}}{(z_j-1)^s z_j^r}\,.$$

Note the squared Vandermonde determinant.

Saddle point equation and Random Matrices $(\Delta = 0)$

We can view MIR as a Random Matrix Model with logarithmic potential (Triple Penner Model):

$$F_N(r,s) = \frac{(-1)^{s(s+1)/2}}{s!2^{s^2(1/y-1)}(2\pi i)^s} \oint_{C_0} \cdots \oint_{C_0} d^s z \exp\left\{ \sum_{\substack{j,k=1 \\ j\neq k}}^{s} \ln|z_j - z_k| \right.$$

$$\left. + s\sum_{j=1}^{s} \left[\left(\frac{1}{y} - 1\right) \ln(z_j+1) - \ln(z_j-1) - \frac{x}{y}\ln(z_j) \right] \right\}.$$

Saddle Point Equation (SPE) reads:

$$\frac{1}{z_j - 1} + \frac{x/y}{z_j} - \frac{(1/y-1)}{z_j+1} = \frac{2}{s}\sum_{k=1}^{s} \frac{1}{z_j - z_k}, \qquad j = 1,2,\ldots,s$$

There are many standard approaches developed for Random Matrix models, to solve such saddle-point eq. These methods are of course applicable here too, (this has been done) but in the present case they turn out to be rather involved, and anyway thay cannot be generalized to the case of generic $\Delta \neq 0$.

Even in the $\Delta = 0$, this is rather complicate. But we do not need the full solution!

A simple exercise: $s = 1$

$$F_N(r,1) = -\frac{1}{2^{N-1}} \oint_{C_0} \frac{(z+1)^{N-1}}{(z-1)z^r}\, dz$$

Large N behaviour: $\quad x = \dfrac{r}{N}$ fixed.

Solution of saddle point equation is: $\quad z_{sp} = \dfrac{x}{1-x}$

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Large N behaviour: $x = \dfrac{r}{N}$ fixed.

Solution of saddle point equation is: $z_{sp} = \dfrac{x}{1-x}$

- When $z_{sp} < 1$ $(0 < x < \frac{1}{2})$ we get: $F_N(x) \sim \mathrm{e}^{-Nf(x)}$
- When $z_{sp} > 1$ $(\frac{1}{2} < x < 1)$ we get: $F_N(x) \sim \underbrace{-\mathrm{Res}_{z=1}}_{=1!} + \mathrm{e}^{-Nf(x)}$

As $N \to \infty$ we get a step function behavior.

The step occurs when x is such that $z_{sp} = 1$: $\Longrightarrow$ $\Theta(x - 1/2)$

This mechanism holds for any finite value of s.

A nice identity

The following identity holds:

$$\frac{(-1)^{s(s+1)/2}}{s!(t^2+1)^{s(N-s)}(2\pi i)^s} \oint_{C_1} \cdots \oint_{C_1} d^s z \prod_{1\le j<k\le s} (z_j - z_k)^2 \prod_{j=1}^{s} \frac{(z_j+1)^{N-s}}{(z_j-1)^s\, z_j^r} = 1\,,$$

Note the different contour C_1: clockwise, encircling $z = 1$, and no other singularity of the integrand.

Single Penner Model

[Penner'88][Ambjorn-Kristjansen-Makeenko'94]

$$Z_N \propto \int \mathcal{D}M \mathrm{e}^{-N\mathrm{Tr}[V(M)]} \qquad M = M^+$$

$$\propto \int d^N z \, \Delta_N^2(z) \, \mathrm{e}^{-N\sum_{j=1}^N V(z_j)} \qquad V(M) = q \, \log M + aM$$

When $q = 1$, the coefficient of $\ln M$ is exactly equal to the order of the Vandermonde. In this case, possibility of `total' condensation of roots of SPEs into the logarithmic well.

Strictly speaking total condensation is impossible (it does not satisfy SPEs). It is to be intended in the sense of condensation of `almost all' roots, but a vanishing fraction.

In the case of `total condensation', among this vanishing fraction of uncondensed roots, there must necessarily be a pair of coinciding real roots.

Summarizing:

- EFP has a step function behaviour in the scaling limit;
- EFP behaviour is governed by the position of SPE roots with respect to the pole at $z = 1$;
- the cumulative residue at such pole is exactly 1;
- Penner model allows for partial/total condensation of eigenvalues in the logarithmic potential well.
- The coefficient of our logarithmic potential well at $z = 1$ is exactly s: possibility of total condensation.

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- EFP has a step function behaviour in the scaling limit;
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- The coefficient of our logarithmic potential well at $z = 1$ is exactly s: possibility of total condensation.

Condensation of \`almost all' SPE roots at $z = 1$ $\Longleftrightarrow$ Arctic Curves

NB: This last statement is in fact a theorem in the Free Fermion case $(\Delta = 0)$

[Colomo-Pronko'07, Bleher-McLaughlin *(to appear)*]

The Arctic curve ($\Delta = 0$)

SPE reads:

$$\frac{1}{z_j - 1} + \frac{x/y}{z_j} - \frac{(1/y - 1)}{z_j + 1} = \frac{2}{s} \sum_{k=1}^{s} \frac{1}{z_j - z_k}.$$

If we assume condensation, in the large s limit $\rho(w) = \delta(w-1)$, and LHS in SPE becomes:

$$\frac{2}{z_j - 1}$$

And the `reduced' SPE thus reads simply

$$-\frac{1}{z_j - 1} + \frac{x/y}{z_j} - \frac{(1/y - 1)}{z_j + 1} = 0,$$

and determines the position of the `very few' possibly uncondensed roots.

We require two coinciding roots:

$$\begin{cases} (x-1)z^2 + (1-2y)z - x = 0 \\ 2(x-1)z + (1-2y) = 0 \end{cases}$$

The solution of the above system (linear in x, y) is

$$x = \frac{1}{z^2+1}, \qquad y = \frac{(z-1)^2}{2(z^2+1)} \qquad z \in [1, +\infty)$$

Which is exactly the parametric form of the (top left quarter of the) Arctic Circle!
Indeed, eliminating z :

$$(x - \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2 = \tfrac{1}{4}$$

Generic values of $\Delta \neq 0$

1) Our nice identity still holds:

$$\left(-\frac{1}{2\pi i}\right)^s \oint_{C_1} \cdots \oint_{C_1} d^s z\, h_{N,s}(z_1,\ldots,z_s) \prod_{j=1}^{s} \frac{1}{z_j^r (z_j-1)^s} \prod_{1\le j<k\le s} \frac{(\tilde{z}_j-1)(z_k-1)(z_j-z_k)}{\tilde{z}_j z_k - 1} = 1$$

2) again the poles at $z_j = 1$ ($j = 1, \ldots, s$) have power s just as the order of the Vandermonde determinant.

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Main assumption

Arctic Curve occurs in correspondence to the following configuration of SPE solutions:

- "almost all" SPE solutions condense to the value $z = 1$;
- a vanishing fraction of SPE solutions survive condensation and lies somewhere in the complex plane; among them there is a pair of coinciding real roots, lying in $[1, +\infty[$.

Generic values of $\Delta \neq 0$

The saddle Point Equation now reads:

$$
\begin{aligned}
-\frac{s}{z_j-1}-\frac{r}{z_j}+s\frac{t^2-2\Delta t}{t^2 z_j-2\Delta t z_j+1}+\partial_{z_j}\ln h_N^{(s)}(z_1,\dots,z_s)& \\
-\frac{t^2-2\Delta t+1}{(t^2 z_j-2\Delta t z_j+1)^2}\partial_{u_j}\ln h_s^{(s)}(u_1,\dots,u_s)+2\sum_{\substack{k=1\\k\neq j}}^{s}\frac{1}{z_j-z_k}& \\
-\sum_{\substack{k=1\\k\neq j}}^{s}\frac{t^2 z_k-2\Delta t}{t^2 z_j z_k-2\Delta t z_j+1}-\sum_{\substack{k=1\\k\neq j}}^{s}\frac{t^2 z_k}{t^2 z_j z_k-2\Delta t z_k+1}&=0,
\end{aligned}
$$

$$
\left(u_j=\frac{1-z_j}{(t^2-2\Delta t)z_j+1}\right)
$$

The procedure of condensation leads to the following equation for the vanishing fraction of uncondensed roots

$$
\frac{y}{z-1}-\frac{x}{z}-\frac{yt^2}{t^2 z-2\Delta t+1}+\lim_{N\to\infty}\frac{1}{N}\partial_z \ln h_N(z)=0.
$$

The $\Delta = 1/2$ case (ASMs)

In this case the explicit form of $h_N(z)$ is known for any N [Zeilberger'95]:

$$h_N(z) = {}_2F_1\left(\begin{matrix} -N+1,\ N \\ 2N \end{matrix}\middle| 1-z\right)$$

Using Euler integral representation, we readily evaluate its large N behaviour:

$$\ln h_N(z) = N \ln\left[4v(1-v)(1-v+zv)\right] + o(N),$$

where

$$v := \frac{2-z-\sqrt{z^2-z+1}}{3(1-z)}.$$

The reduced SPE now reads:

$$\frac{y}{z-1} - \frac{1-x+y}{z} + \frac{1-\sqrt{z^2-z+1}}{z(1-z)} = 0.$$

Requiring two coinciding roots we obtain:

$$(2x-1)^2 + (2y-1)^2 - 4xy = 1\,, \qquad x, y \in \left[0, \tfrac{1}{2}\right].$$

`Limit shape' of ASMs

The $\Delta = -\frac{1}{2}$ can be treated analogously.

Generic values of Δ (disordered regime $|\Delta| < 1$)

We come back to our fundamental equation for the vanishing fraction of uncondensed roots:

$$\frac{y}{z-1} - \frac{x}{z} - \frac{yt^2}{t^2 z - 2\Delta t + 1} + \lim_{N\to\infty} \frac{1}{N} \partial_z \ln h_N(z) = 0\,.$$

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We need now the large N behaviour of $h_N(z)$, for generic Δ:

$$h_N(z) \underset{N\to\infty}{\sim} \left[\frac{\sin\gamma(\lambda-\eta)}{\gamma \sin(\lambda-\eta)}\right]^N \left[\frac{\sin(\zeta+\lambda-\eta)\sin(\gamma\zeta)}{\sin\gamma(\zeta+\lambda-\eta)\sin\zeta}\right]^N e^{o(N)}$$

where

$$z(\zeta) = \frac{\sin(\lambda+\eta)}{\sin(\lambda-\eta)} \frac{\sin(\zeta+\lambda-\eta)}{\sin(\zeta+\lambda+\eta)} \,, \qquad \text{and} \qquad \gamma := \frac{\pi}{\pi - 2\eta} \,.$$

$$\Delta = \cos 2\eta \qquad\qquad t = \frac{\sin(\lambda-\eta)}{\sin(\lambda+\eta)}$$

NB: $z \in [1, +\infty)$ corresponds to $\xi \in [0, \pi - \lambda - \eta)$

The equation for uncondensed roots now read:

$$x\Phi(\zeta+\lambda-\eta,2\eta)-y\Phi(\zeta,2\eta)+\Phi(\zeta,\lambda-\eta)-\gamma\Phi(\gamma\zeta,\gamma(\lambda-\eta))=0$$

where

$$\Phi(\mu,\nu)=\frac{\sin(\nu)}{\sin(\mu)\sin(\mu+\nu)}\,.$$

Its derivative is:

$$x\Psi(\zeta+\lambda-\eta,2\eta)-y\Psi(\zeta,2\eta)+\Psi(\zeta,\lambda-\eta)-\gamma^2\Psi(\gamma\zeta,\gamma(\lambda-\eta))=0\,,$$

where

$$\Psi(\mu,\nu)=\frac{\sin\nu\sin(2\mu+\nu)}{\sin^2\mu\sin^2(\mu+\nu)}\,.$$

Solve the above system, linear in x, y:

We get:

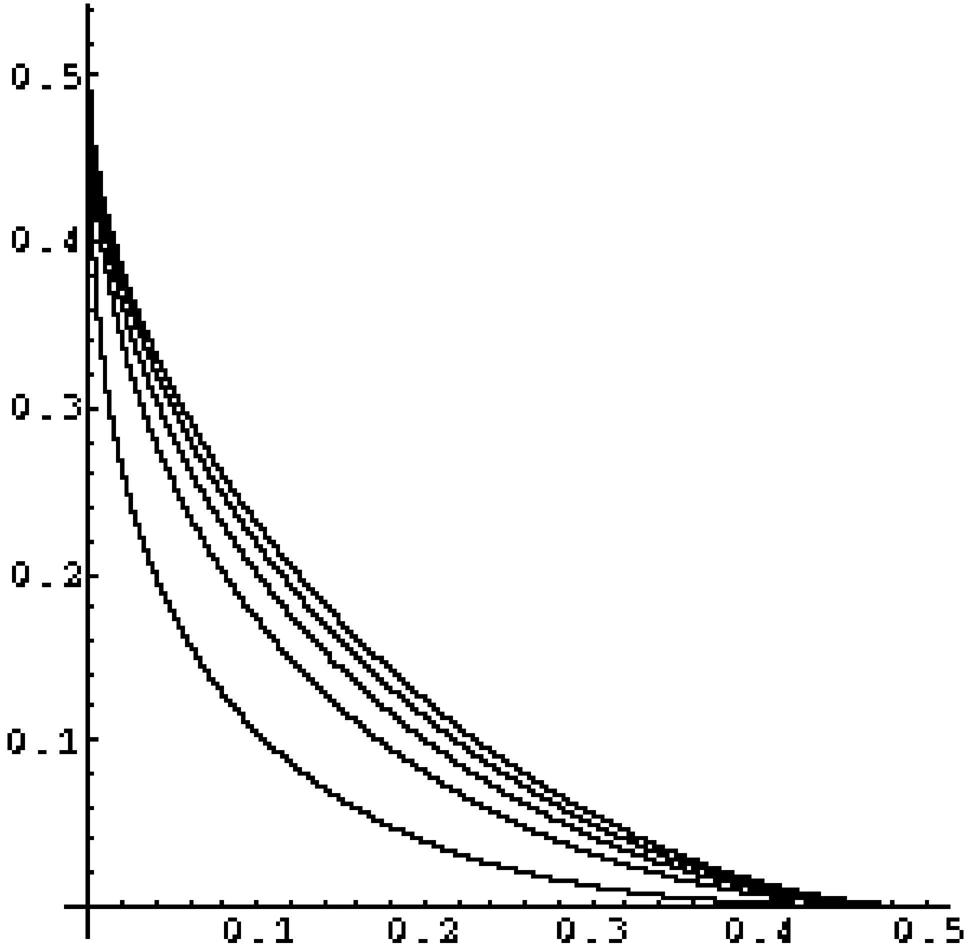
$$
\begin{aligned}
x &= \frac{1}{\Phi(\zeta+\lambda-\eta,2\eta)\Psi(\zeta,2\eta)-\Psi(\zeta+\lambda-\eta,2\eta)\Phi(\zeta,2\eta)} \\
&\qquad\qquad \times\left\{\left[\Psi(\zeta,\lambda-\eta)-\gamma^2\Psi(\gamma\zeta,\gamma(\lambda-\eta))\right]\Phi(\zeta,2\eta)\right. \\
&\qquad\qquad\qquad \left.-\left[\Phi(\zeta,\lambda-\eta)-\gamma\Phi(\gamma\zeta,\gamma(\lambda-\eta))\right]\Psi(\zeta,2\eta)\right\}, \\
y &= \frac{1}{\Phi(\zeta+\lambda-\eta,2\eta)\Psi(\zeta,2\eta)-\Psi(\zeta+\lambda-\eta,2\eta)\Phi(\zeta,2\eta)} \\
&\qquad \times\left\{\left[\Psi(\zeta,\lambda-\eta)-\gamma^2\Psi(\gamma\zeta,\gamma(\lambda-\eta))\right]\Phi(\zeta+\lambda-\eta,2\eta)\right. \\
&\qquad\qquad \left.-\left[\Phi(\zeta,\lambda-\eta)-\gamma\Phi(\gamma\zeta,\gamma(\lambda-\eta))\right]\Psi(\zeta+\lambda-\eta,2\eta)\right\}.
\end{aligned}
$$

Parametric form of limit shape for generic Δ, with parameter $\zeta\in[0.\,\pi-\lambda-\eta]$, and

$$
\gamma := \frac{\pi}{\pi-2\eta}, \qquad \Phi(\mu,\nu)=\frac{\sin(\nu)}{\sin(\mu)\sin(\mu+\nu)}, \quad \Psi(\mu,\nu)=\frac{\sin\nu\sin(2\mu+\nu)}{\sin^2\mu\sin^2(\mu+\nu)}.
$$

NB: γ rational $\Longrightarrow$ algebraic curve

γ irrational $\Longrightarrow$ non-algebraic curve



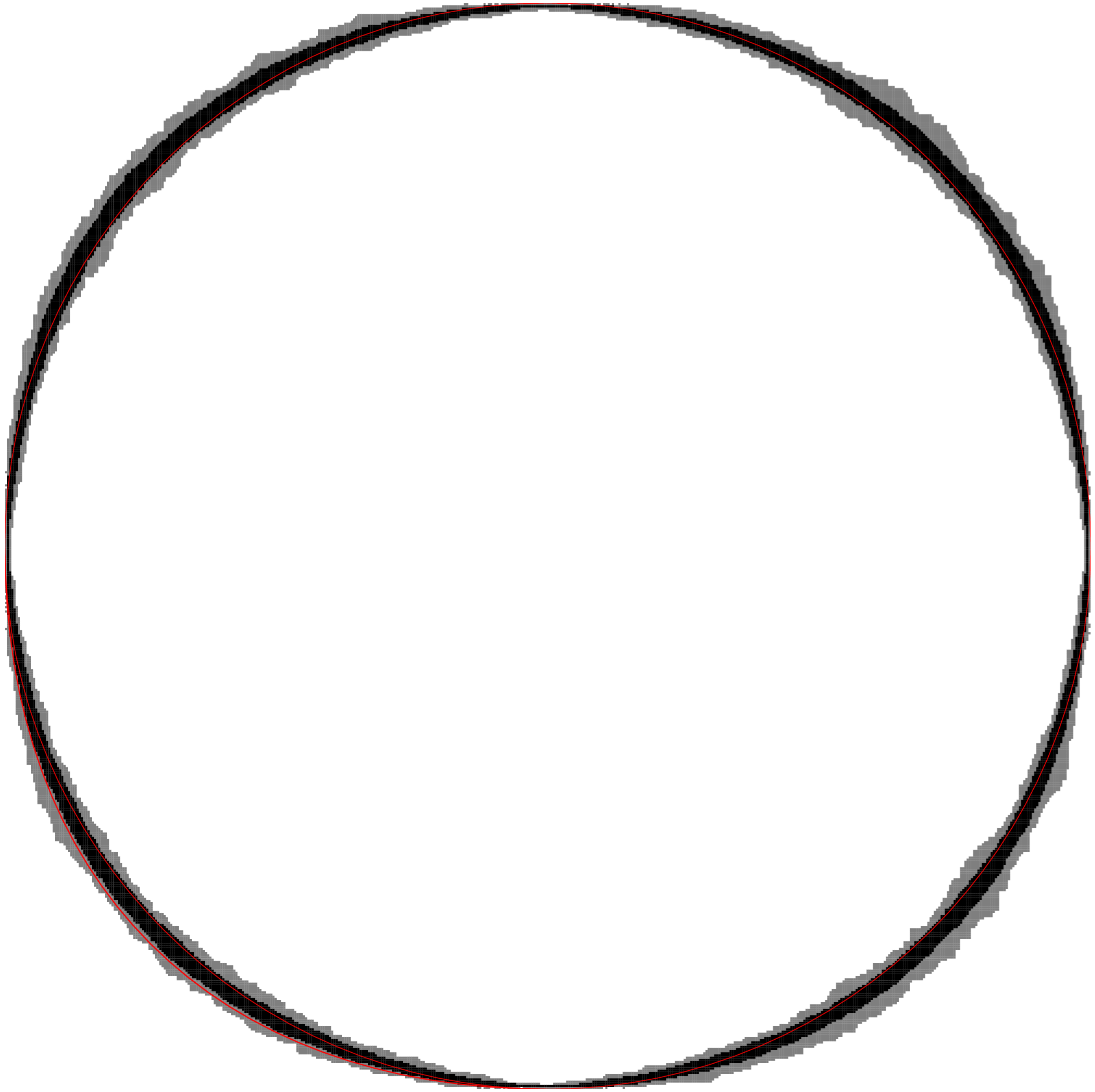
Limit shapes for $\Delta = 0.9,\ 0.5,\ 0,\ -0.5,\ -0.9.$

$$(2x-1)^2 + (2y-1)^2 - 4xy = 1, \qquad x, y \in \left[0, \tfrac{1}{2}\right].$$

ASMs: N=500

199 samples

$\Delta = 1/2$



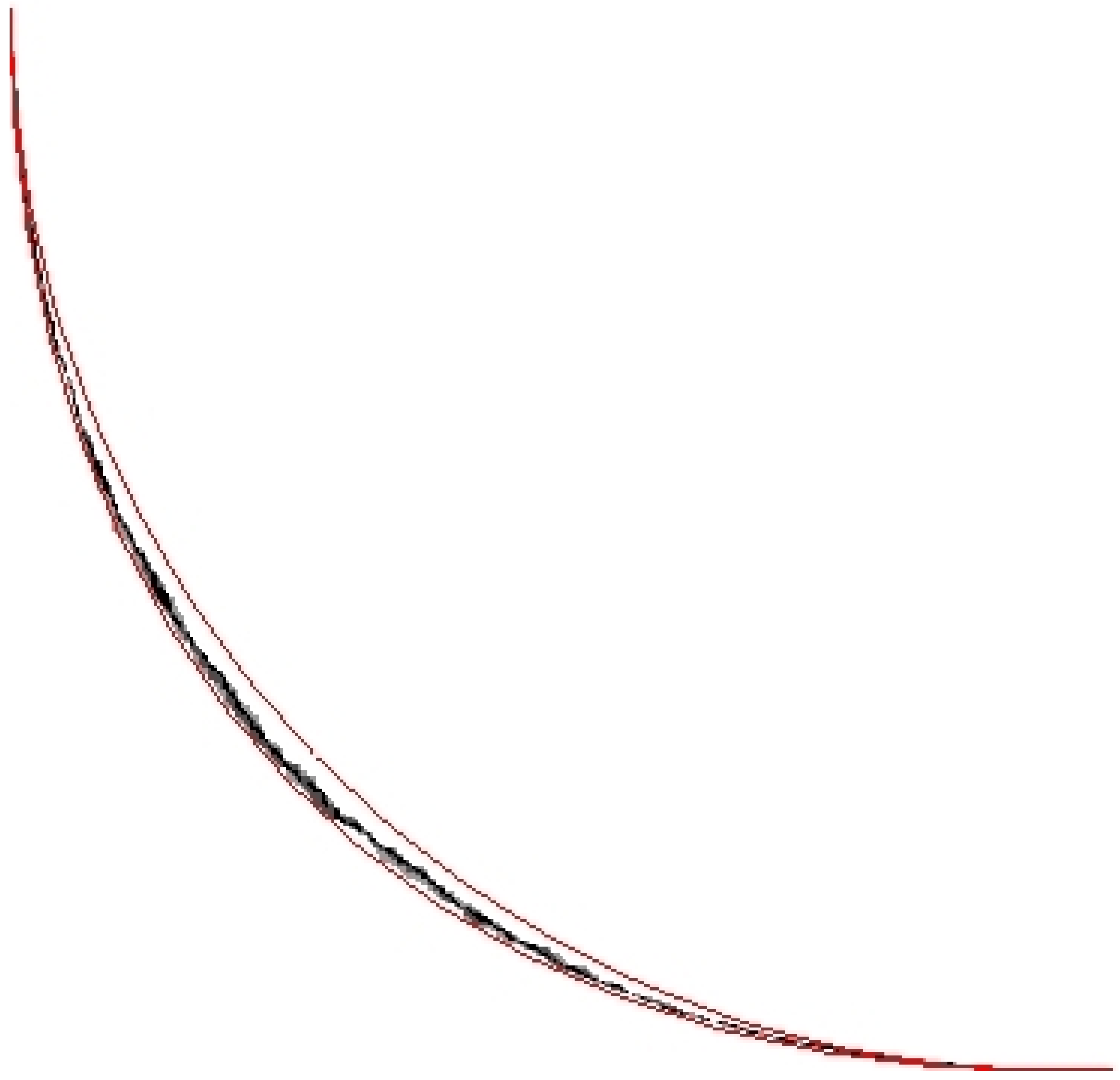
Ben Wieland (January 2008)

http://www.math.brown.edu/~wieland

ASMs: N=1500

10 samples

$\Delta = 1/2$



Ben Wieland (April 2008)

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What about the $q \rightarrow 0$ limit?

For finite N, ASMs $\xrightarrow[q\to 0]{}$ `permutation matrices'.

We expect a trivial arctic curve, degenerating to the boundary of the unit square

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From general formula we get instead:

$$x + y = \frac{1}{2} - \frac{1}{\pi}\cos\pi(x-y)\,, \qquad\qquad x, y \in [0, \frac{1}{2}]$$

NB: $N \to \infty$ and $q \to 0$ do not commute.

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Question: What does this curve describe?
Of which model is it the Arctic curve?

Final comments

- Fluctuations of the limit shape are driven by the evaporation of SPE solutions from the logarithmic well (Penner potential of Random Matrices), just like in the $\Delta = 0$ case. From universality considerations, the Airy process of Arctic Circle [Johansson'05] is again expected.

- Some MIR for correlations function in ASEP has been obtained [Tracy-Widom'07], which strongly remind our ones for DW 6VM. “Condensation Ansatz” could play a role there too?

- The last step of our derivation is not valid in the (physically more interesting) AF regime, $\Delta < -1$. Work is in progress.