Problèmes inverses à la frontière pour l'équation de Beltrami dans des domaines plans, approximation dans des classes de Hardy généralisées

Juliette Leblond

INRIA SOPHIA ANTIPOLIS projet APICS

joint work with

L. Baratchart, S. Rigat, E. Russ

INRIA, Sophia-Antipolis, LATP-CMI, Univ. Provence, U. Paul Cézanne, Marseille

Conductivity equation

Let $\Omega \subset \mathbb{R}^2$ smooth and $\sigma \in C(\bar{\Omega})$, $0 < c \leq \sigma \leq C$

$$\text{div } (\sigma \nabla u) = 0 \text{ in } \Omega \tag{1}$$

- Cauchy problems: $|I|\,,\ |\partial\Omega \setminus I| > 0$

$$\text{tr } u \text{ and } \partial_n u \text{ prescribed on } I \subset \partial\Omega$$

$$\text{recover } u \text{ in } \Omega \text{ and Cauchy data on } J = \partial\Omega \setminus I$$

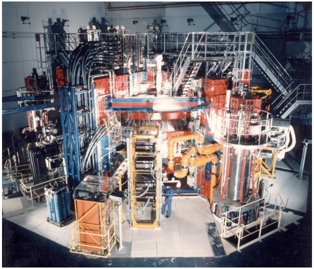
- Dirichlet problem:

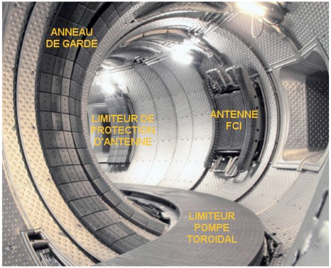
$$\text{tr } u \text{ prescribed on } \partial\Omega$$

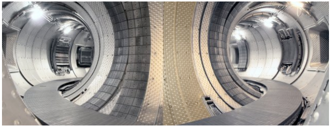
$$\text{recover } u \text{ in } \Omega \text{ and } \partial_n u \text{ on } \partial\Omega$$

A motivation...

Recover shape of plasma boundary in a tokamak



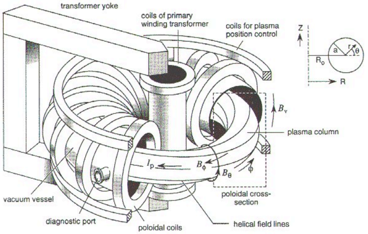


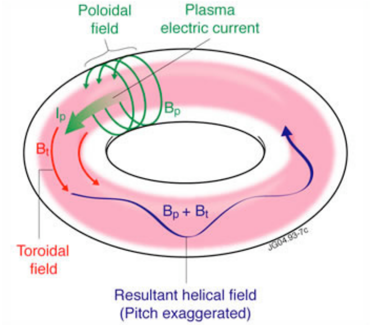


Tore Supra (CEA-IRFM Cadarache)

... A motivation...

Maxwell equations, cylindrical coordinates (x, y, ϕ) of magnetic induction, axial symmetry (indep. of ϕ)





... A motivation

$\rightsquigarrow$ in poloidal section (annular domain) $(x, y) \in \Omega \subset \mathbb{R}^2$
poloidal magnetic induction [Bl]:

$$B = \begin{bmatrix} B_x \\ B_y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \sigma \nabla u \quad , \quad \text{conductivity } \sigma = \frac{1}{x}$$

for poloidal magnetic flux u:

$$\text{div } (\sigma \nabla u) = 0 \text{ in } \Omega$$

given u and $B \approx \sigma \partial_n u$ on $I \subset \partial\Omega$

look for u and $\partial_n u$ on $\partial\Omega \setminus I$? level line of u (plasma boundary)?

Conjugated ($\mathbb{R}$-linear) Beltrami equation

u solution to (1): $\operatorname{div}(\sigma\nabla u) = 0$ iff

$$u = \operatorname{Re} f$$

where $f = f(z, \bar{z})$ satisfies first order elliptic equation

$$\bar{\partial} f = \nu \overline{\partial f} \text{ in } \Omega \qquad (2)$$

with respect to complex variable $z = x + iy$ and [AP]

$$\nu = \frac{1-\sigma}{1+\sigma}$$

$\nu \in C(\bar{\Omega})$ real-valued, $|\nu| \leq \kappa < 1$ in Ω

$\mathbb{C}$-linear Beltrami equation: $\bar{\partial} g = \nu \partial g$ quasi-conformal map. [Ahlf., Ast.]

Generalized σ-harmonic conjugation

we have

$$f = u + i\,v$$

where v σ-harmonic conjugated function

Hilbert-Riesz transform

$$\operatorname{div}\left(\frac{1}{\sigma}\nabla v\right) = 0 \text{ in } \Omega$$

unique up to additive constant

generalized Cauchy-Riemann equations in Ω:

$$\nabla v = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \sigma \nabla u \;:\; \begin{cases} \partial_x v = -\sigma \partial_y u \\ \partial_y v = \sigma \partial_x u \end{cases}$$

Proof

$$\partial = \partial_z = \frac{1}{2}(\partial_x - i\,\partial_y)\,,\ \ \bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\,\partial_y)$$

....

generalization of (σ constant)

$$\Delta u = 0 \ (u \text{ harmonic}) \Leftrightarrow \bar{\partial} f = 0 \ (f \text{ analytic}) \text{ in } \Omega$$

Smooth solutions to Dirichlet problem

Thm [Campanato] $1 < p < \infty$

$\forall \phi \in W^{1-1/p,p}_{\mathbb{R}}(\partial\Omega)$, there exists $f \in W^{1,p}(\Omega)$ solution to (2) in Ω such that $\mathrm{Re}\,\mathrm{tr}\; f = \phi$ on $\partial\Omega$

$$\text{unique if normalization condition } \int_{\partial\Omega} \mathrm{Im}\,\mathrm{tr}\; f\, d\theta = 0 \qquad (3)$$

$$\text{further} \quad \|f\|_{W^{1,p}(\Omega)} \leq C\, \|\varphi\|_{W^{1-1/p,p}(\partial\Omega)}$$

$u = \mathrm{Re}\, f \in W^{1,p}(\Omega)$, $u = \phi$ on $\partial\Omega$ unique solution to (1)

(in $W^{1,2}(\Omega)$ Lax-Milgram - also for $\sigma \in L^\infty(\Omega)$ - in $W^{2,p}(\Omega)$ [ADN]; for $\sigma \in \mathrm{VMO}(\Omega)$ [AQ])

allows to solve boundary approximation problems but with Sobolev norms and smooth boundary data

With $L^p(\partial\Omega)$ boundary data?

$\Omega = \mathbb{D}$ unit disk, $L^p(\mathbb{T})$ data

simply connected Ω

smooth σ, $\nu \in W^{1,\infty}(\mathbb{D})$

Generalized Hardy spaces $H^p_\nu = H^p_\nu(\mathbb{D})$ of solutions:
functions f on $\mathbb{D}$ satisfying

$\mathbb{T}_r$ circle radius r

$$\|f\|_{H^p_\nu} = \operatorname*{ess\,sup}_{0<r<1} \|f\|_{L^p(\mathbb{T}_r)} < +\infty$$

solutions to (2) in $\mathbb{D}$ as distributions

$$(\|f\|^p_{L^p(\mathbb{T}_r)} = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta)$$

$H^p_\nu \subset L^p(\mathbb{D})$ real Banach space

Harmonic and analytic functions

$\sigma \equiv 1$ (cst), $\Delta u = 0$ in $\mathbb{D}$ $(\nu = 0)$

classical Hardy spaces $H^p = H^p_0(\mathbb{D})$ of analytic functions

$$\bar{\partial} f = 0 \text{ and } \|f\|_{H^p} < +\infty$$

$\mathrm{f} = \mathrm{u} + \mathrm{i}\,\mathrm{v}$, conjugated function v: $\Delta v = 0$ in $\mathbb{D}$ Hilbert-Riesz transform

Cauchy-Riemann equations:

$$\begin{cases} \partial_x v = -\partial_y u \text{ in } \mathbb{D} \\ \partial_y v = \partial_x u \end{cases} \quad \begin{cases} \partial_n v = -\partial_\theta u \text{ on } \mathbb{T} \\ \partial_\theta v = \partial_n u \end{cases}$$

Hardy spaces H^p

- Properties of H^p Banach spaces (below...)
- Poisson-Cauchy-Green representation formulas, analytic projection
- Hilbert H^2, Fourier basis:

$$H^2 = \{\sum_{n\geq 0} f_n z^n \, , \, \sum_{n\geq 0} |f_n|^2\} \text{ , tr } H^2 : z = e^{i\theta} \in \mathbb{T}$$

- allow to state and solve above issues as best approximation problems on $L^p(I)$ or $L^p(\mathbb{T})$ [BL]

Properties of H^p_ν...

Generalize those of H^p

- Fatou:

$$\|\text{tr } f\|_{L^p(\mathbb{T})} \leq \|f\|_{H^p_\nu} \leq c_\nu \, \|\text{tr } f\|_{L^p(\mathbb{T})}$$

$$\lim_{r\to 1} \int_0^{2\pi} \left| f(re^{i\theta}) - \text{tr } f(e^{i\theta}) \right|^p d\theta = 0$$

- tr H^p_ν closed subspace of $L^p(\mathbb{T})$

If $f \in H^p_\nu$:

- $\log |\text{tr } f| \in L^1(\mathbb{T})$ (does not vanish on positive measure subsets) unless $f \equiv 0$ in $\mathbb{D}$
- If $f \not\equiv 0$, then its zeros α_j are isolated in $\mathbb{D}$

$$\sum_{j=1}^{\infty}(1 - |\alpha_j|) < +\infty \qquad \text{(with multiplicity)}$$

... Properties of H^p_ν

Let $H^{p,0}_\nu \subset H^p_\nu$ of f such that (3) holds

- If $f \in H^{p,0}_\nu$ is such that $\mathrm{Re}\,(\mathrm{tr}\, f) = 0$ a.e. on $\mathbb{T}$, then $f \equiv 0$ in $\mathbb{D}$
- If $f \in W^{1,p}(\mathbb{D})$ solution of (2), then $f \in H^p_\nu$ with

$$\|f\|_{H^p_\nu} \leq C_{\nu,p} \|f\|_{W^{1,p}(\mathbb{D})}$$

+ orthogonal space and duality

Density results

Thm $I \subset \mathbb{T}$ measurable subset, $|\mathbb{T} \setminus I| > 0$

- the space of restrictions to I of functions in tr H^p_ν is dense in $L^p(I)$
- tr H^p_ν weakly closed in $L^p(\mathbb{T})$
- let $(f_k)_{k\geq 1} \in H^p_\nu$ whose trace on I converges to ϕ in $L^p(I)$:
  either ϕ is already the trace on I of an H^p_ν function
  or $\|\text{tr } f_k\|_{L^p(\mathbb{T}\setminus I)} \to +\infty$

$\rightsquigarrow$ bounded approximation problems (BEP)

if $I = \text{Int } \bar{I} \neq \mathbb{T}$ (in particular, I is open), the space of restrictions to I of traces on $\mathbb{T}$ of solutions to (CB) in $W^{1,p}(\mathbb{D})$ is dense in $W^{1-1/p,p}(I)$

Dirichlet theorem

Thm For all $\varphi \in L^p_{\mathbb{R}}(\mathbb{T})$, $\exists$ unique $f \in H^{p,0}_\nu$ such that a.e. on $\mathbb{T}$:

$$\mathrm{Re}\,(\mathrm{tr}\ f) = \varphi$$

moreover $\|f\|_{H^p_\nu} \leq c_{p,\nu} \|\varphi\|_{L^p(\mathbb{T})}$

hence, Hilbert transform (conjugation op.) continuous $L^p(\mathbb{T})$:

$$\mathrm{Re}\,(\mathrm{tr}\ f) = u_{|_\mathbb{T}} = \varphi \overset{\mathcal{H}_\nu}{\mapsto} \mathrm{Im}\,(\mathrm{tr}\ f) = v_{|_\mathbb{T}} = \mathcal{H}_\nu(\varphi)$$

+ higher regularity results, $\mathcal{H}_\nu$ ctn on $W^{1-1/p,p}(\mathbb{T})$ then $W^{1,p}(\mathbb{T})$

Dirichlet-Neumann map $\Lambda_\sigma = \partial_\theta \mathcal{H}_\nu$ [AP], Calderón

Tools and ideas of the proof...

Thm [BN] Let $\alpha \in L^\infty(\mathbb{D})$

$$\alpha = -\frac{\bar\partial \nu}{1-\nu^2} = \frac{\bar\partial \sigma}{2\sigma} = \bar\partial \log \sigma^{1/2}$$

$$f = u + i\,v \in H^p_\nu \Longleftrightarrow w = \frac{f - \nu \bar f}{\sqrt{1-\nu^2}} = \sigma^{1/2}\, u + i\,\sigma^{-1/2}\, v \in G^p_\alpha$$

Hardy spaces of solutions to

$$\bar\partial w = \alpha \overline{w} \tag{4}$$

$$f \in W^{1,p}(\mathbb{D}) \text{ solves (2)} \Leftrightarrow w \in W^{1,p}(\mathbb{D}) \text{ solves (4)}$$

$$G^{p,0}_\alpha : \text{ with normaliz. cond. } \int_{\mathbb{T}} \sigma^{1/2}\, \mathrm{Im\,tr}\;\, w\, d\theta = 0$$

w solves Schrödinger equ.

... Tools and ideas of the proof...

Thm [BN] Every $w \in G^p_\alpha$ admits in $\mathbb{D}$ a representation

$$w = e^s F$$

for $s \in W^{1,q}(\mathbb{D})$, $\forall q \in (1, +\infty)$ and $F \in H^p$

further $\quad \|s\|_{L^\infty(\mathbb{D})} \leq c\|\alpha\|_{L^\infty(\mathbb{D})}$

s can be chosen such that $\mathrm{Re}\, s = 0$ on $\mathbb{T}$ (or $\mathrm{Im}\, s = 0$)

(hence $s \in C^{0,\gamma}(\overline{\mathbb{D}})$, $\forall \gamma \in (0,1)$; also $w \in W^{1,q}_{loc}(\mathbb{D})$, $\forall q \in (1, +\infty)$)

Proof: take $r = \alpha \overline{w}/w$ if $w \neq 0$ ($r = 0$ if $w = 0$) and $\bar{\partial} s = r$ in $\mathbb{D}$

... Tools and ideas of the proof...

Properties of G^p_α: similar to H^p_ν non tangential limit, Fatou, uniqueness

Representation: for $w \in G^p_\alpha(\mathbb{D})$ and a.e. $z \in \mathbb{D}$ Cauchy-Green formula

$$w(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\text{tr } w(\xi)}{\xi - z} d\xi + \frac{1}{2\pi i} \iint_{\mathbb{D}} \frac{\alpha \overline{w(\xi)}}{\xi - z} d\,\xi \wedge d\,\overline{\xi}$$

whence $w = \mathcal{C}\,(\text{tr } w) + T_\alpha w$

boundedness properties of Cauchy operators $\mathcal{C}$, T_α

$$w = (I - T_\alpha)^{-1} \mathcal{C}\,(\text{tr } w)$$

Re (tr w) $\mapsto$ tr w continuous on $G^{p,0}_\alpha$

... Tools and ideas of the proof

Dirichlet: for all $\varphi \in L^p_{\mathbb{R}}(\mathbb{T})$, $\exists$ unique $w \in G^{p,0}_\alpha$ such that

$$\mathrm{Re}\,(\mathrm{tr}\ w) = \varphi \quad \text{a.e. on } \mathbb{T}$$

$$\text{moreover } \|w\|_{G^p_\alpha} \leq c_{p,\nu}\,\|\varphi\|_{L^p(\mathbb{T})}$$

For H^p fos: for all $\varphi \in H^p$, $\exists$ unique $w \in G^{p,0}_\alpha$ such that

$$P_+(\mathrm{tr}\ w) = \mathrm{tr}\ \varphi \quad \text{a.e. on } \mathbb{T}$$

a.e. in $\mathbb{D}$, $w = \varphi + T_\alpha(w)$; moreover

$$\|w\|_{G^p_\alpha} \leq C_p\,\|\varphi\|_{H^p}$$

Back to conductivity equation

- solve Dirichlet problem with given data $\phi = u_{|_{\mathbb{T}}}$ in $L^p(\mathbb{T})$
- Cauchy type issues? on I, from (noisy) data, get

$$f = u_{|_I} u + i \int (\sigma \partial_n) u_{|_I} \text{ in } L^p(I) \text{ but } f \notin (\text{tr } H^p_\nu)_{|_I}$$

in view of density:

$$\inf_{h \in \text{tr } H^p_\nu} \|f - h\|_{L^p(I)} = 0$$

while for such a sequence h_n, $\|h_n\|_{L^p(\mathbb{T} \setminus I)} \nearrow \infty$

Bounded extremal problems

However, with norm constraint $M > 0$ the BEP:

$$\min_{\substack{h \in \mathrm{tr}\ H^p_\nu \\ \|h\|_{L^p(\mathbb{T}\setminus I)} \le M}} \|f - h\|_{L^p(I)}$$

achieved by a unique $h_0 \in \mathrm{tr}\ H^p_\nu$ such that $\|h_0\|_{L^p(\mathbb{T}\setminus I)} \le M$
further, if $f \notin (\mathrm{tr}\ H^p_\nu)_{|I}$

$$\|h_0\|_{L^p(\mathbb{T}\setminus I)} = M$$

$\rightsquigarrow$ solution to Cauchy problem

Further results in H^p_ν

- orthogonal space and duality
- higher regularity results

Conclusion...

- simply connected smooth Ω: conformal mapping $\psi : \mathbb{D} \to \Omega$

$$\bar{\partial}(f \circ \psi) = (\nu \circ \psi)\,\overline{\partial(f \circ \psi)}$$

- annulus $\mathbb{A} = \mathbb{D} \setminus \varrho\overline{\mathbb{D}}$: annular domains

$\overline{H^p_\nu}(\varrho\mathbb{D})$: Hardy space of solutions to (2) in $\mathbb{C} \setminus \varrho\bar{\mathbb{D}}$

$$H^p_\nu(\mathbb{A}) = H^p_{\nu_i}(\mathbb{D}) \oplus \overline{H^p_{\nu_e}}(\varrho\mathbb{D})$$

$\nu_i \in W^{1,\infty}(\mathbb{D})$, $\nu_e \in W^{1,\infty}(\mathbb{C} \setminus \varrho\overline{\mathbb{D}})$ such that $\nu_{i|_{\mathbb{A}}} = \nu_{e|_{\mathbb{A}}} = \nu$

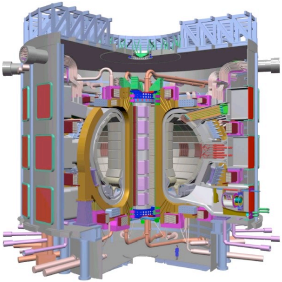
as for classical Hardy spaces, with $\nu, \nu_i, \nu_e = 0$

- related PDEs Schrödinger ($\Delta w = aw + b\bar{w}$ in $\mathbb{C}$), Laplace ($\Delta U = 0$ in $\mathbb{R}^3$)

... Conclusion...

- computation of solutions to (BEP) for $p = 2$:
  $\mathcal{H}_\nu$? $\rightsquigarrow \perp$ projection $L^p(\mathbb{T}) \to$ tr H^p_ν? in H^2
- application to plasma/tokamaks: $\sigma(x, y) = 1/(x + x_0)$
- bases of H^2_ν? families of Bessel functions?

$$\nu(z, \bar{z}) = \frac{z + \bar{z} + 2x_0 - 2}{z + \bar{z} + 2x_0 + 2}$$

- geometrical issues: free boundary Bernoulli pb?
-  ITER

Main references

[Ahl] Ahlfors, Lectures on quasiconformal mappings (1966)

[AP] Astala, Päivärinta, Calderòn inverse conductivity pb in the plane (2006)

[BLRR] Baratchart, Leblond, Rigat, Russ, Hardy spaces of the conjugate Beltrami equation (in preparation)

[BN] Bers, Nirenberg, On a representation theorem for linear elliptic systems with discontinuous coefficients and its applications (1954)

[LV] Lehto, Virtanen, Quasiconformal mappings in the plane (1973)

[V] Vekua, Generalized analytic functions (1962)

......