A Non-Holonomic Systems Approach to Special Function Identities

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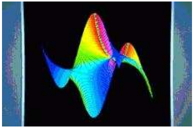

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I Introduction

Computer Algebra & Special-Function Identities

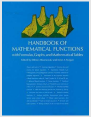
Goal

Symbolic manipulation of expressions and automatic proof of identities involving special functions, combinatorial sequences, etc.

Linear differential/recurrence equations are a good data structure!

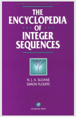
- Zeilberger's *Holonomic systems approach* (1990).
- Equations defeat closed-form expressions: BS & Zimmermann (1994 + gfun), FC & BS (1998), FC (2000), etc.
- Meunier's ESF & the DDMF by ABe, FC, SG & MM.

An ubiquitous class



About 25% of Sloane's encyclopedia,
60% of Abramowitz & Stegun.

eqn + initial cond. = data structure



Examples: Sums & Integrals

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$$ [Strehl92]

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1)\binom{2n}{n}^2$$ [Blodgelt90]

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x)\, dx = -\frac{\ln(1-a^4)}{2\pi a^2}$$ [GlMo94]

$$\frac{1}{2\pi i} \oint \frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\, dy = \frac{H_n(x)}{\lfloor n/2 \rfloor !}$$ [Doetsch30]

$$\int_0^{\infty} \int_0^{\infty} J_1(x) J_1(y) J_2(c\sqrt{xy}) \frac{dx\, dy}{e^{x+y}}$$ satisfy a 2nd-order ODE

+ many, many more in, e.g., .

Examples: q-Sums, Integral Transforms & Symmetric Functions

$$\sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}} \quad \text{[Andrews74]}$$

$$\sum_{j=0}^{n} \sum_{i=0}^{n-j} \frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j}(q;q)_i(q;q)_j} = \sum_{k=-n}^{n} \frac{(-1)^k q^{7/2k^2+1/2k}}{(q;q)_{n+k}(q;q)_{n-k}} \quad \text{[Paule85]}$$

$$\int_{-1}^{+1} \frac{e^{-px} T_n(x)}{\sqrt{1-x^2}}\,dx = (-1)^n \pi I_n(p)$$

$$\int_0^{+\infty} x e^{-px^2} J_n(bx) I_n(cx)\,dx = \frac{1}{2p} \exp\left(\frac{c^2-b^2}{4p}\right) J_n\left(\frac{bc}{2p}\right)$$

$$\left\langle \exp\big((p_1^2 - p_2)/2 - p_2^2/4\big) \,\middle|\, \exp\big(t(p_1^2+p_2)/2\big) \right\rangle = \frac{e^{-\frac{1}{4}t(t+2)}}{\sqrt{1-t}}$$

Examples: Non-"Holonomic" Sums & Integrals

$$\sum_{k=0}^{n} \binom{n}{k} i(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n \quad \text{[Abel1826]}$$

$$\sum_{k=0}^{n} (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ {n+1 \atop k+1} \right\} = \left\langle {n \atop m} \right\rangle \quad \text{[Frobenius1910]}$$

$$\sum_{k=0}^{m} \binom{m}{k} B_{n+k} = (-1)^{m+n} \sum_{k=0}^{n} \binom{n}{k} B_{m+k} \quad \text{[Gessel03]}$$

$$\int_0^\infty x^{k-1} \zeta(n, \alpha + \beta x)\, dx = \beta^{-k} B(k, n-k) \zeta(n-k, \alpha)$$

$$\int_0^\infty x^{\alpha-1} \operatorname{Li}_n(-xy)\, dx = \frac{\pi(-\alpha)^n y^{-\alpha}}{\sin(\alpha\pi)}$$

$$\int_0^\infty x^{s-1} \exp(xy) \Gamma(a, xy)\, dx = \frac{\pi y^{-s}}{\sin\big((a+s)\pi\big)} \frac{\Gamma(s)}{\Gamma(1-a)}$$

Two Ideas — Several Generations of Algorithms

Ideas

- Confinement in finite-dimensional vector-spaces
  $\longrightarrow$ closures under algebraic operations
- Creative telescoping ($\approx$ differentiation under the integral sign)
  $\longrightarrow$ closures under $\int$, $\sum$, $\langle \dots | \dots \rangle$

Classes of inputs

- (q-)hypergeometric/hyperexponential: Zeilberger (1990), Paule–Schorn (1995); Riese (2003); Almkvist–Z. (1990)
- higher-order eqns: Zeilberger (1990), Takayama (1989–90), FC–BS (1998), FC (2000)
- Abel-type/Stirling/Euler and Bernoulli: Majewicz (1996); Kauers (2007); Chen & Sun (2009)
- previous and more: the present work

II Algebraic Closures

Before the examples, recall . . .

Gauss' hypergeometric series,

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| x\right) := \sum_{n=0}^{\infty} u_n x^n \quad \text{for} \quad u_n := \frac{(a)_n (b)_n}{(c)_n n!}, \quad \text{solves} $$

$$ (\theta + c - 1)\theta(F) = x(\theta + a)(\theta + b)(F) \quad \text{for } \theta := x\,\frac{d}{dx}, $$

$$ \text{since} \quad \frac{u_{n+1}}{u_n} = \frac{(n+a)(n+b)}{(n+c)(n+1)}. $$

$$ \theta^2(F) \in \langle F, \theta(F) \rangle \quad (\text{over } \mathbb{Q}(a,b,c)(x)) $$

This generalizes to $\quad {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n n!} x^n.$

Algebraic Closures by Vector-Space Confinement (1)

Example of a product of generating series: Clausen's identity,

$$ {}_2F_1\left(\begin{matrix} a, b \\ a+b+1/2 \end{matrix}\middle| x\right)^2 = {}_3F_2\left(\begin{matrix} 2a, 2b, a+b \\ 2(a+b), a+b+1/2 \end{matrix}\middle| x\right). $$

Proof: After setting $f = g^2$,

1. $f \in \langle g^2 \rangle, \qquad \theta(f) = 2g\theta(g) \in \langle g\theta(g) \rangle,$
   $\theta^2(f) \in \langle \theta(g)^2, g\theta^2(g) \rangle \subset \langle g^2, g\theta(g), \theta(g)^2 \rangle =: V,$
   $\theta^3(f) \in \langle g^2, g\theta(g), \theta(g)^2, g\theta^2(g), \theta(g)\theta^2(g) \rangle \subset V.$
2. 4 vectors in dim. 3 $\rightarrow$ 3rd-order ODE for f (confinement).
3. Same ODE for the ${}_3F_2$ + match up to $O(x^4)$ + Cauchy thm.

Algebraic Closures by Vector-Space Confinement (2)

Example of the generating series of a product: Mehler's identity for Hermite polynomials,

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}.$$

Proof:

1. Define Hermite polynomials $H_n(t)$ (D-finite over $\mathbb{Q}(t)(n)$): recurrence of order 2;
2. For the product, introduce the vector basis over $\mathbb{Q}(x,y)(n)$

$$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$

$\rightarrow$ recurrence of order at most 4; (confinement)

3. Translate into differential equation and solve.

Multivariate Confinement Implies Several Relations

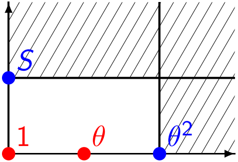
Example: Contiguity relation (Gauss, 1812) for hypergeometric series,

$$F(a,x) := {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| x\right) := \sum_{n=0}^{\infty} u_{a,n} x^n, \qquad u_{a,n} := \frac{(a)_n (b)_n}{(c)_n n!}.$$

$$\frac{u_{a,n+1}}{u_{a,n}} = \frac{(a+n)(b+n)}{(c+n)(n+1)} \to (1-x)\theta^2(F) = ((a+b)x + 1 - c)\theta(F) + abxF,$$

$$\frac{u_{a+1,n}}{u_{a,n}} = \frac{n}{a} + 1 \to S(F) := F(a+1,x) = \frac{1}{a}\theta(F) + F.$$

dim. $= 2 \Rightarrow S^2(F), S(F), F$ dependent over $\mathbb{Q}(b,c)(a,z)$:



$$(a+1)(x-1)S^2(F) + ((b-a-1)x + 2 - c + 2a)S(F) + (c-a-1)F = 0$$

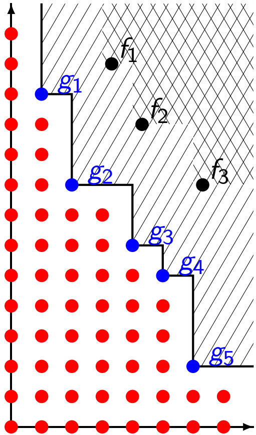
Crash Course on Gröbner Bases

Generalize Euclidean division and the Euclidean (gcd) algorithm

1. Monomial ordering: total order $\prec$ on the monomials, compatible with the product, with minimal element 1
2. Gröbner basis of a (left) ideal I w.r.t. $\prec$: generators matching the stairs of I

$$\mathrm{GB}(f_1, f_2, f_3) = (g_1, \dots, g_5)$$

3. Quotient: vector basis below the stairs
4. Reduction of P modulo I: unique remainder written on this basis



Dimension of Ideals and ∂-Finite Functions

$M_k(I) := \langle\, m : m$ is below the stairs and of total degree $\leq s \,\rangle$

Theorem (Hilbert)

For any I, there is an integer $\delta(I)$ such that $\#M_k(I) = O\big(s^{\delta(I)}\big)$.

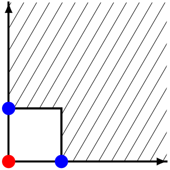
- $\delta(I)$ is the (Hilbert) dimension of I.
- Finite measure of infinite-dimensional vector-spaces.
- Can be obtained by a Gröbner-basis calculation.
- $I \subset J \Rightarrow \delta(I) \geq \delta(J)$

Definition (annihilator and ∂-finiteness)

- $\operatorname{ann} f \subset S = \{\, P \in S : P(f) = 0 \,\}$
- f is ∂-finite $\Leftrightarrow$ linear dim. of quotient is finite $\Leftrightarrow$ $\delta(\operatorname{ann} f) = 0$

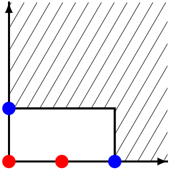
Examples

Binomial coeffs $\binom{n}{k}$ w.r.t. S_n, S_k; Hypergeometric sequences:



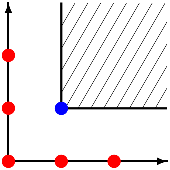
$\delta(I) = 0$, $\dim S/I = 1$

Bessel $J_\nu(x)$ w.r.t. S_ν, D_x; Orthogonal polys w.r.t. S_n, D_x:



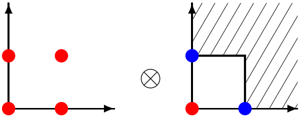
$\delta(I) = 0$, $\dim S/I = 2$

Stirling nbs w.r.t. S_n, S_k:



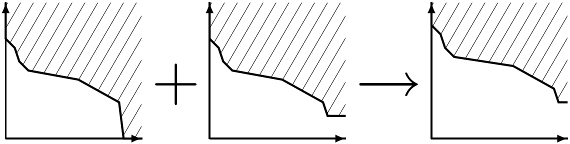
$\delta(I) = 1$, $\dim S/I = \infty$

Abel-type w.r.t. S_m, S_k, S_r, S_s, $h(m,k)(k+r)^k(m-k+s)^{m-k}\frac{r}{k+r}$:



$\delta(I) = 2$ in space of dimension 4

Closure Properties



Proposition

$$\delta(\operatorname{ann}(f+g)) \le \max\bigl(\delta(\operatorname{ann} f), \delta(\operatorname{ann} g)\bigr),$$
$$\delta(\operatorname{ann} fg) \le \delta(\operatorname{ann} f) + \delta(\operatorname{ann} g), \quad \delta(\operatorname{ann} \partial f) \le \delta(\operatorname{ann} f).$$

Algorithm (for a product fg): for a *graded* ordering,

for $s = 0, 1, 2, \dots$, until $\delta(I) \le$ bound:
  for each $|\alpha| \le s$, reduce $\partial^\alpha(fg)$ to a sum $\sum u_{\alpha;\beta,\gamma}(x)\partial^\beta(f)\partial^\gamma(g)$
    over $\beta \in M_s(\operatorname{ann} f), \gamma \in M_s(\operatorname{ann} g)$
  search for $Q(x)$-linear relations, set I to the ideal they generate
return I, a subideal of $\operatorname{ann} fg$

Product of Stirling Numbers of the Second Kind

$\left\{{n \atop k}\right\}$ and $\left\{{m \atop k}\right\}$, given by $\delta(I_1) = \delta(I_2) = 1$:

$$\left\{{n+1 \atop k+1}\right\} = (k+1)\left\{{n \atop k+1}\right\} - \left\{{n \atop k}\right\}, \quad \Delta_m\left\{{n \atop k}\right\} = 0,$$

$$\Delta_n\left\{{m \atop k}\right\} = 0, \quad \left\{{m+1 \atop k+1}\right\} = (k+1)\left\{{m \atop k+1}\right\} - \left\{{m \atop k}\right\}.$$

Computations up to $s = 3$: $\binom{6}{3} = 20 > 19 = \dim M_3(I)$.

$u_{n,m,k} := \left\{{n \atop k}\right\}\left\{{m \atop k}\right\}$, given by $\delta(I) = 2$:

$$u_{n+1,m+1,k+1} - (k+1)u_{n+1,m,k+1} - (k+1)u_{n,m+1,k+1} + (k+1)^2 u_{n,m,k+1} - u_{n,m,k} = 0.$$

III Closures under $\sum$ and $\int$

Summation by Creative Telescoping

$$\text{Goal: evaluate } S_n := \sum_{k=0}^{n} \binom{n}{k} \text{ to } 2^n.$$

Given Pascal's triangle rule,

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1} = 2\binom{n}{k} + \binom{n}{k+1} - \binom{n}{k},$$

summation over k yields $S_{n+1} = 2S_n$.
The initial condition $S_0 = 1$ concludes the proof.

Creative Telescoping for Sums (Zeilberger, 1990)

$$U_n = \sum_{k=a}^{b} u_{n,k} = ?$$

Given $A(n, S_n)$ and $B(n, k, S_n, S_k)$ such that

$$\big(A(n, S_n) + \Delta_k B(n, k, S_n, S_k)\big)(u)(n, k) = 0,$$

summation leads by "telescoping" to

$$A(n, S_n)(U)(n) = \big[B(n, k, S_n, S_k)(u)(n, k)\big]_{k=a}^{k=b+1} \overset{\text{often}}{=} 0.$$

Adapts easily to $U(x) = \sum_{k=a}^{b} u_k(x)$.

Creative Telescoping for Integrals (Zeilberger, 1990)

$$U(x) = \int_a^b u(x,y)\,dy = ?$$

Given $A(x, D_x)$ and $B(x, y, D_x, D_y)$ such that

$$\big(A(x, D_x) + D_y B(x, y, D_x, D_y)\big)(u)(x,y) = 0,$$

integrating leads by “telescoping” to

$$A(x, D_x)(U)(x) = \big[B(x, y, D_x, D_y)(u)(x,y)\big]_{y=a}^{y=b+1} \overset{\text{often}}{=} 0.$$

Adapts easily to $U_n = \int_a^b u_n(y)\,dy$.

Towards Algorithms for Creative Telescoping

Example: $U(x) := \int_0^1 \frac{\cos xy}{\sqrt{1-y^2}} dy = ?$

$$U'(x) = \int_0^1 -y \frac{\sin xy}{\sqrt{1-y^2}} dy,$$

$$U''(x) = \int_0^1 -y^2 \frac{\cos xy}{\sqrt{1-y^2}} dy = -U(x) + \int_0^1 \sqrt{1-y^2} \cos xy \, dy,$$

$$U''(x) + U(x) = \left[\sqrt{1-y^2} \frac{\sin xy}{x} \right]_{y=0}^{y=1} + \int_0^1 \frac{y}{\sqrt{1-y^2}} \frac{\sin xy}{x} \, dy = -\frac{U'(x)}{x}.$$

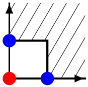
$$U''(x) + \frac{U'(x)}{x} + U(x) = \left[-\frac{1-y^2}{xy} D_x(u)(x,y) \right]_{y=0}^{y=1} = 0.$$

$$D_x^2 + \frac{1}{x} D_x + 1 + D_y \frac{1-y^2}{xy} D_x \in \operatorname{ann} u \quad \to \quad U(x) = J_0(x).$$

Creative telescoping = finite difference under the sum + summation by parts
Creative telescoping = differentiation under the integral + integration by parts

Example: Rediscovering Pascal's Triangle Rule

Reduce all monomials of degree $\leq s = 2$:



$$1 \to 1, \quad S_n \to \frac{n+1}{n+1-k}\,1, \quad S_k \to \frac{n-k}{k+1}\,1$$

$$S_n^2 \to \frac{(n+2)(n+1)}{(n+2-k)(n+1-k)}\,1, \quad S_k^2 \to \frac{(n-k-1)(n-k)}{(k+2)(k+1)}\,1, \quad S_nS_k \to \frac{n+1}{k+1}\,1.$$

Common denominator: $D_2 = (k+1)(k+2)(n+1-k)(n+2-k)$.

$D_2, D_2S_n, D_2S_k, D_2S_n^2, D_2S_k^2, D_2S_nS_k$ confined in

$$\langle 1, k\,1, k^2\,1, k^3\,1, k^4\,1 \rangle \text{ over } K(n)$$

$$\to \quad D_2(S_nS_k - S_k - 1) \in \operatorname{ann}\binom{n}{k}.$$

$\deg D_s = O(s) \Rightarrow$ this had to happen for some s.

More Examples

- $\dfrac{1}{n+k}$: essentially the same situation.

- $\dfrac{1}{n^2+k^2}$: confinement in a space of dimension $O(s^2)$,
  no elimination of k succeeds.

- $f = \dfrac{a(x, y_1, \dots, y_r)}{b(x, y_1, \dots, y_r)}$: $D_s = b^s$,
  confinement in a space of dimension $O(s^r)$ over $K(x)$,
  elimination of $y_1, \dots, y_r$ has to succeed.
  Base case of proof that D-finite functions are "holonomic".

Adding a Left Part and a Right Part

Specification of a Creative-Telescoping Algorithm

Input: generators of (a subideal of) ann f

Output: all (A, B) such that:

- $A + \partial_y B \in \operatorname{ann} f$,
- A is free of y and ∂_y

Algorithms for specific classes. Often return truncated results.

Definition (telescoping ideal of I w.r.t. y)

$$T_y(I) := \left(I + \partial_y S_{x,y}\right) \cap S_x \quad \text{where}$$

$$S_{x,y} := K(x,y)\langle \partial_x, \partial_y \rangle \text{ and } S_x := K(x)\langle \partial_x \rangle.$$

When $I = S_{x,y} G_1 + \cdots + S_{x,y} G_\ell$, this involves

$$(S_{x,y} G_1 + \cdots + S_{x,y} G_\ell) + \partial_y S_{x,y}.$$

Polynomial Growth and Creative Telescoping when $\delta > 0$

Definition (polynomial growth p)

There exists a sequence of polynomials $P_s(x, y)$, s.t. if $|a| + b \leq s$,

$$P_s \partial_{x_1}^{a_1} \dots \partial_{x_k}^{a_k} \partial_y^b \text{ reduces to polys of degree } O(s^p) \text{ in } y.$$

Theorem (Chyzak, Kauers & Salvy, 2009)

$$\delta(T_y(I)) \leq \max\big(\delta(I) + p - 1, 0\big).$$

Proof: $\quad P_s \partial_x^a \partial_y^b - \sum_{\beta \in M_s(I)} \big(\text{deg. in } y \leq O(s^p)\big) \partial^\beta \in I$

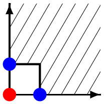
$\Rightarrow$ any choice $\partial_{x_1}, \dots, \partial_{x_{\delta(I)+p}}, \partial_y$ is algebraically dependent modulo $I \Rightarrow \delta(T_y(I)) \leq \delta(I) + p - 1$.

Corollary (sufficient condition for creative telescoping)

$\delta(I) + p - 1 < k \Rightarrow$ *identities exist for the sum/int. w.r.t. y.*

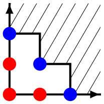
Examples with Polynomial Growth $p = 1$

- Proper hypergeometric (Wilf & Zeilberger, 1992):

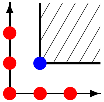


$$Q(n,k)\xi^k \frac{\prod_{i=1}^{u} (a_i n + b_i k + c_i)!}{\prod_{i=1}^{v} (u_i n + v_i k + w_i)!},$$

Q polynomial, a_i, b_i, u_i, v_i integers.



- Differential D-finite (special case of "holonomy").
- Stirling: $\delta = 1 \to$ for ≥ 3 vars, e.g., Frobenius:



$$\sum_{k=0}^{n} (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ {n+1 \atop k+1} \right\} = \left\langle {n \atop m} \right\rangle .$$

- Abel type: $\delta = 2 \to$ for ≥ 4 vars, e.g., Abel:

$$\sum_{k=0}^{n} \binom{n}{k} i(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n.$$

Algorithm I: Fasenmyer's Style

Polynomial growth + linear algebra $\rightarrow J := I \cap S_x[\partial_y]$.

Algorithm

For increasing values of s, until $\delta(J) \leq$ bound:

1. Reduce all $\partial_x^a \partial_y^b$ with $a + b \leq s$;
2. Normalize to a common denominator;
3. Set up a linear system to cancel the positive powers of y;
4. If a non-zero solution is found, it has the form $A(x, \partial_x) + \partial_y B(y, \partial_x, \partial_y)$; return it.

This computes A in

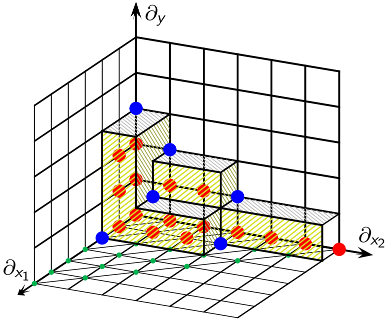
$$(J + \partial_y S_x[\partial_y]) \cap S_x, \quad \text{not in} \quad T_t(I) := (I + \partial_y S_{x,y}) \cap S_x.$$

$$S_x = K(x)\langle \partial_x \rangle \subset S_x[\partial_y] \subset S_{x,y} := K(x,y)\langle \partial_x, \partial_y \rangle.$$

Algorithm II. Zeilberger's Style Extended to Dim > 0

Compute A in $T_t(I) := (I + \partial_y S_{x,y}) \cap S_x$. $\leftrightarrow$ faster, more precise.

1. (q-)Hypergeometric case: Zeilberger 1990 (+ Schorn, Riese).
2. ∂-finite case ($\delta = 0$): Chyzak 2000 (+ Koutschan, Pech).
3. Non-∂-finite: new.



for $s = 0, 1, 2, \dots,$ until $\delta(J) \leq$ bound:
  set $A := \sum_{|\alpha| \leq s} \eta_\alpha \partial^\alpha$
    for undetermined coeffs $\eta_\alpha(y) \in K(x)$
  set $B := \sum_{\beta \in M_s(I)} \phi_\beta(y) \partial^\beta$
    for undetermined coeffs $\phi_\beta(y) \in K(x, y)$
  reduce $A - \partial_y B$
    onto the basis $M_{s+1}(I)$
  extract coeffs to form a linear system
    of first order w.r.t. ∂_y
  solve and set J to the ideal of the A's
return the pairs (A, B)

IV Conclusion

Conclusion

- Summary:
  - Linear differential/recurrence equations as a data structure;
  - Confinement in vector spaces + creative telescoping $\rightarrow$ identities;
  - Input dimension + polynomial growth $\rightarrow$ output dimension.
- Also:
  - Multiple summation/integration;
  - Bounds $\rightarrow$ identities + their size + complexity of algorithms.
- Open questions:
  - Replace polynomial growth by something intrinsic;
  - Exploit symmetries;
  - Structured Padé-Hermite approximants;
  - Understand non-minimality.

Special-Functions Memento

$J_n(x)$, $Y_n(x)$, $I_n(x)$, $K_n(x)$ (Bessel fn), $H_n(x)$ (Hermite), $T_n(x)$ (Chebyshev) : EDO + rec.

$$\Gamma(s,x) = \text{(upper) incomplete gamma fn} = \int_x^\infty t^{s-1}e^{-t}dt$$

$$B(s,t) = \text{beta fn} = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \qquad \zeta(s,x) = \text{Hurwitz's zeta fn} = \sum_{n=0}^\infty \frac{1}{(n+x)^s}$$

$$(x)_n = \text{Pochhammer symbol} = x(x+1)\dots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}$$

$$(x;q)_n = q\text{-Pochhammer symbol} = (1-x)(1-qx)\dots(1-q^{n-1}x)$$

$$\mathrm{Li}_n(z) = \text{polylogarithms} = \sum_{k=1}^\infty \frac{z^k}{k^n} = \theta_z(\mathrm{Li}_{n+1})(z)$$

$$B_n = \text{Bernoulli nb} = \frac{1}{2i\pi}\oint \frac{1}{e^z-1}\frac{n!}{z^n}dz \qquad E_n = \text{Euler nb} = \frac{1}{2i\pi}\oint \frac{2e^z}{e^{2z}+1}\frac{n!}{z^n}dz$$

$$\left\langle {n \atop m} \right\rangle = \text{Eulerian nb} : \left\langle {n+1 \atop m+1} \right\rangle = (m+2)\left\langle {n+1 \atop m} \right\rangle + (n-m)\left\langle {n \atop m} \right\rangle$$

$$\left[{n \atop m} \right] = \text{Stirling nb of the 1st kind} = \left[{n+1 \atop m+1} \right] = \left[{n \atop m} \right] - (n+1)\left[{n \atop m+1} \right]$$

$$\left\{ {n \atop m} \right\} = \text{Stirling nb of the 2nd kind} = \left\{ {n+1 \atop m+1} \right\} = (m+1)\left\{ {n \atop m+1} \right\} + \left\{ {n \atop m} \right\}$$