Geodesics in large planar quadrangulations

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arXiv:0712.2160, arXiv:0805.2355 and work in progress

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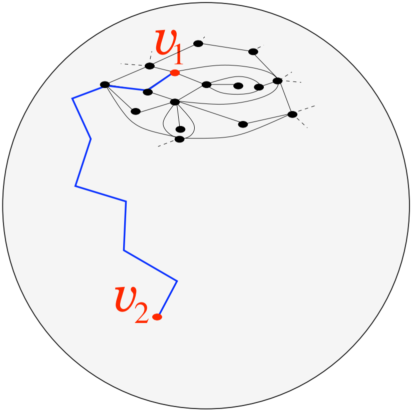
Outline

Introduction

Reminder : geodesic = shortest path between two points



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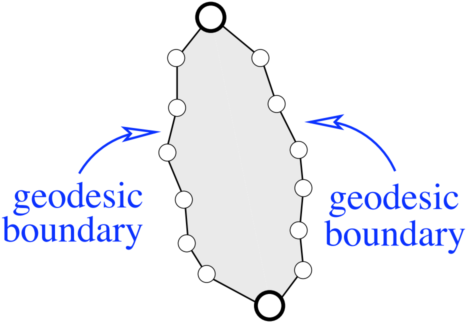
Quadrangulations with geodesic boundary



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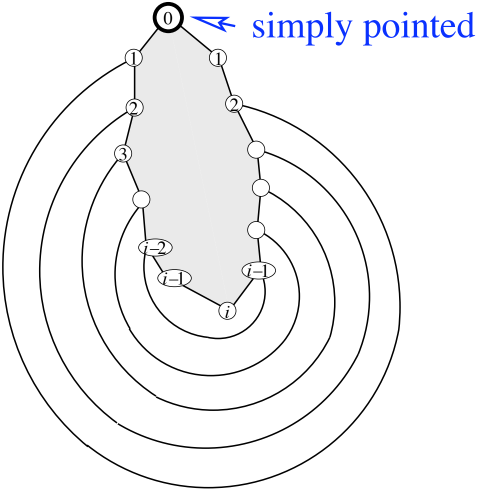

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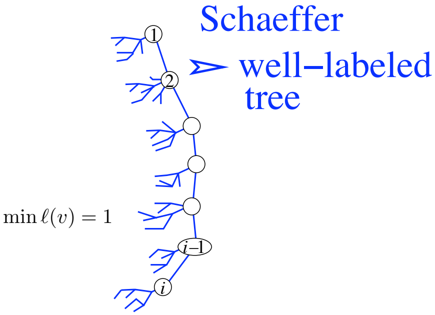

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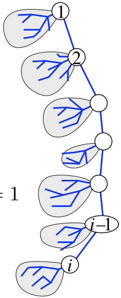

$\min \ell(v) = 1$

$$\prod_{m=1}^{i} R_m$$

Quadrangulations with geodesic boundary

The generating function for quadrangulations with geodesic boundary is therefore:

$$Z_i(g) = \prod_{j=1}^{i} R_j = R^i \frac{(1-x)(1-x^{i+3})}{(1-x^3)(1-x^{i+1})}$$

Reminder: g weight per square,

$$R(g) = \frac{1-\sqrt{1-12g}}{6g} \qquad x(g) + \frac{1}{x(g)} + 1 = \frac{1}{gR(g)^2}$$

Quadrangulations with a marked geodesic

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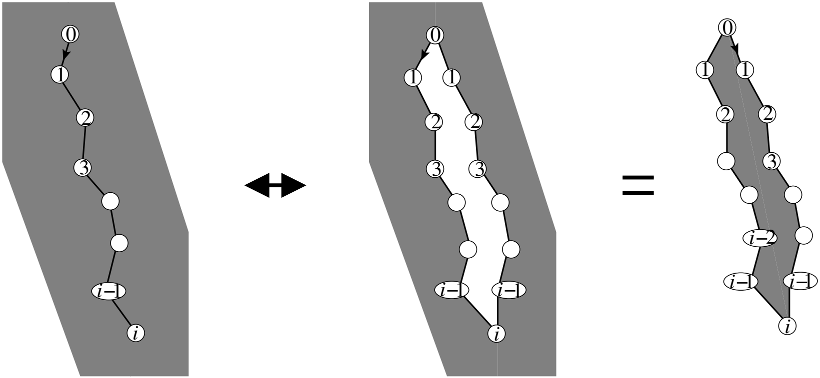
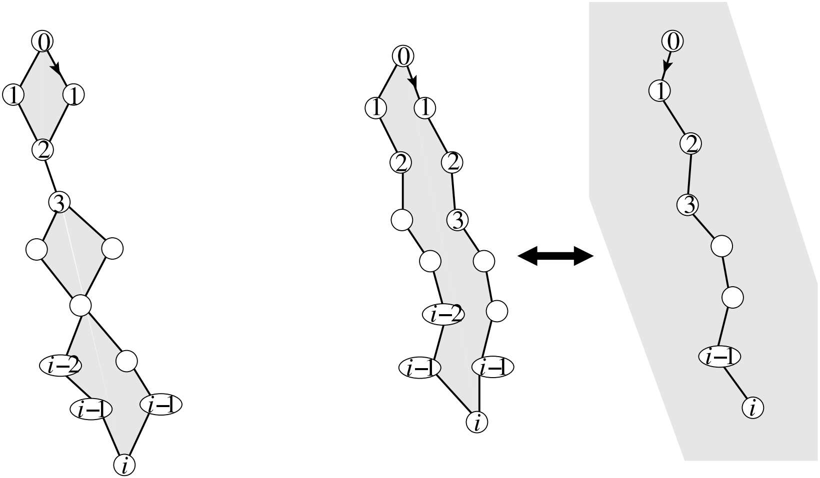
Almost the same as quadrangulations with geodesic boundary...



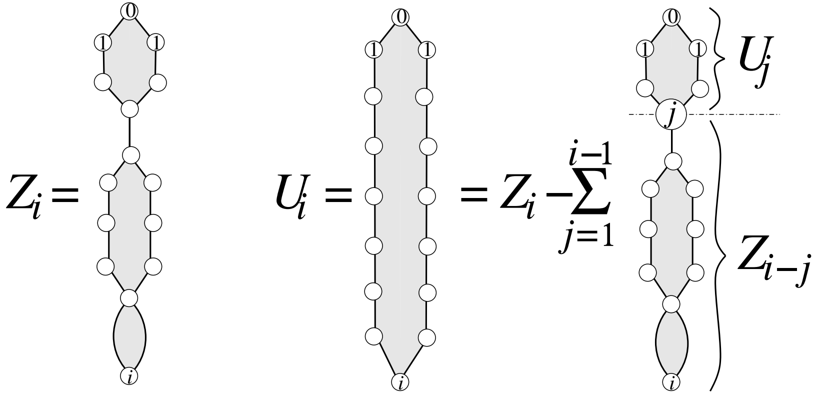
Quadrangulations with a marked geodesic

Arbitrary geodesic boundaries may have "pinch points".
Marked geodesics correspond to irreducible boundaries.



Quadrangulations with a marked geodesic

An arbitrary geodesic boundary may be decomposed into irreducible components.



Quadrangulations with a marked geodesic

$$U_i(g) = Z_i(g) - \sum_{j=1}^{i-1} U_j(g) Z_{i-j}(g) \quad \text{i.e.} \quad \hat{U}(g;t) = \frac{\hat{Z}(g;t)}{1+\hat{Z}(g;t)}$$

From the exact formula for Z_i we can perform asymptotic analysis:

$$U_i(g)\big|_{g^n} \sim \frac{12^n}{2\sqrt{\pi}n^{5/2}} \delta_i \quad \text{as } n \to \infty$$

where:

$$\hat{\delta}(t) = \frac{3t\left(2t\left(3+177t-412t^2+708t^3-624t^4+224t^5\right)+3(1-2t)^6\log(1-2t)\right)}{70(1-2t)^4\left(t-(1-2t)\log(1-2t)\right)^2}$$

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where:

$$\delta_i \sim \frac{9}{7} 2^i i^3 \quad \text{as } i \to \infty$$

Quadrangulations with a marked geodesic

In the local limit:

$$U_i(g)\big|_{g^n} \sim \frac{12^n}{2\sqrt{\pi}n^{5/2}} \times \frac{3}{7} \cdot i^3 \times 3 \cdot 2^i$$

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A similar result holds in the scaling limit $i = r \cdot n^{1/4}$:

$$U_i(g)|_{g^n} \sim \frac{12^n}{2\sqrt{\pi}n^{7/4}} \times \rho(r) \times 3 \cdot 2^i$$

$\rho(r)$: canonical two-point function

Geodesic watermelons

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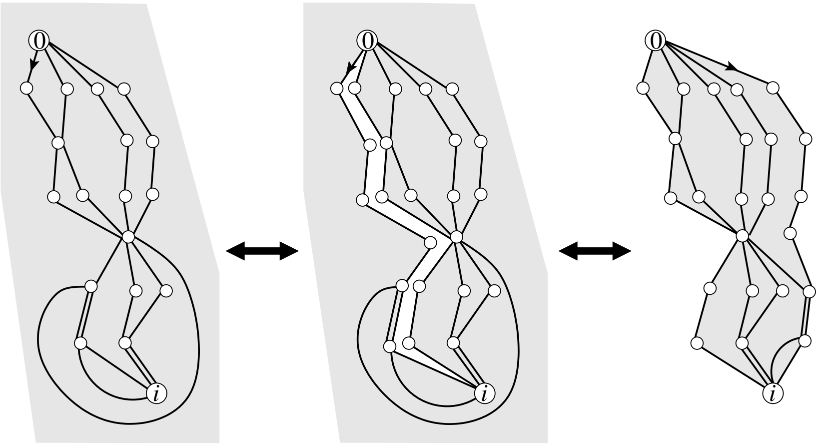

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$$U_i^{(k)} = (Z_i)^k - \sum_{j=1}^{i-1} U_j^{(k)} (Z_{i-j})^k$$

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- Strongly avoiding case: each part must be irreducible

$$\tilde{U}_i^{(k)} = (U_i)^k$$

Geodesic watermelons

In the weakly avoiding case, in the local limit:

$$U_i^{(k)}(g)\Big|_{g^n} \sim \frac{12^n}{2\sqrt{\pi}n^{5/2}} \times \frac{3}{7} \cdot i^3 \times k \cdot \left(3 \cdot 2^i\right)^k$$

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Further computations ($k = 2$):

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In the strongly avoiding case, in the local limit:

$$\tilde{U}_i^{(k)}(g)\Big|_{g^n} \sim \frac{12^n}{2\sqrt{\pi}n^{5/2}} \times \frac{3\cdot 4^{k-1}}{7} i^{6-3k} \times k \cdot (3\cdot 2^i)^k$$

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The constraint of strong avoidance is relevant. In the scaling limit:

$$\tilde{U}_i^{(k)}(g)\Big|_{g^n} \sim \frac{12^n}{2\sqrt{\pi}n^{3k/4+1}} \times \sigma^{(k)}(r) \times k\cdot\left(3\cdot 2^i\right)^k$$

$\sigma^{(k)}(r)$: new scaling functions

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$\sigma^{(k)}(r)$: new scaling functions
Interpretation: only a few exceptional pairs of points can be connected by $k \geq 2$ macroscopically disjoint geodesics. The number of such pairs is of order: $n^{(11-3k)/4}$.

Outline

Geodesic points

Consider a quadrangulation with two marked points (1,2) at distance i. Consider a third point (3) lying on a geodesic between them, say at distance s from 1 (hence $t = i - s$ from 2).

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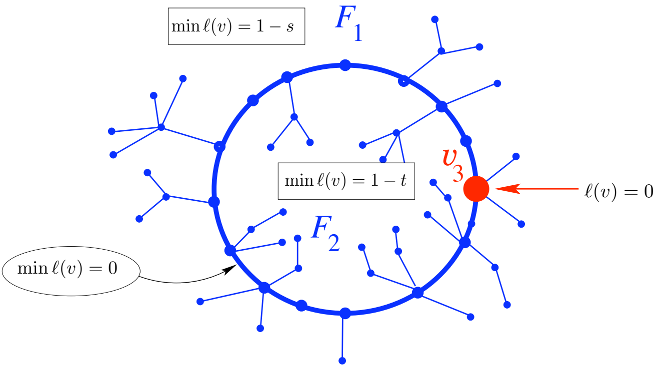
Apply the Miermont bijection with sources 1,2 and delays $\tau_1 = -s$, $\tau_2 = -t$, and obtain a well-labeled map with two faces:



Geodesic points

The generating function for such objects is

$$\Delta_s \Delta_t X_{s,t} = X_{s,t} - X_{s-1,t} - X_{s,t-1} + X_{s-1,t-1}$$

where

$$X_{s,t} = \frac{[3][s+1][t+1][s+t+3]}{[1][s+3][t+3][s+t+1]} \qquad \text{with } [m] \equiv \frac{1-x^m}{1-x}$$

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Upon evaluating $X_{s,t}|_{g^n}$ for $n \to \infty$ and normalizing by the number of quadrangulations with two marked points at distance $i = s + t$, we obtain the mean number of geodesic points:

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$$\langle c(s) \rangle_{s+t} = \frac{1}{N_{s+t}} \Delta_s \Delta_t \xi(s,t)$$

$$\xi(s,t) = \frac{9}{140} \frac{(1+s)(1+t)(3+s+t)}{(3+s)(3+t)(1+s+t)} st\big(29 + 20(s+t) + 5(s^2+t^2+st)\big)(4+s+t)$$

$$N_i = \frac{3}{35}(i+1)(5i^2+10i+2)$$

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$$\langle c(s) \rangle_{s+t} \to \frac{3s(5+s)}{(3+s)(2+s)} \qquad \text{for } t \to \infty$$

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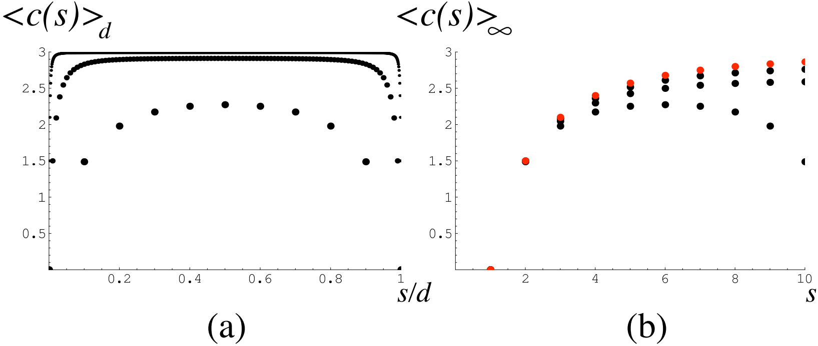
$$\langle c(s) \rangle_{s+t} \to 3 \qquad \text{for } s, t \to \infty$$

Geodesic points



Geodesic points

We actually have access to the full probability law for the number of geodesic points at fixed distances. The g.f. for doubly-pointed quadrangulations with exactly c geodesic points at distances s, t is:

$$\Delta_s \Delta_t X_{s,t}^{(c)} \qquad \text{with} \qquad X_{s,t}^{(c)} = \frac{1}{c} \left(\frac{X_{s,t} - 1}{X_{s,t}} \right)^c$$

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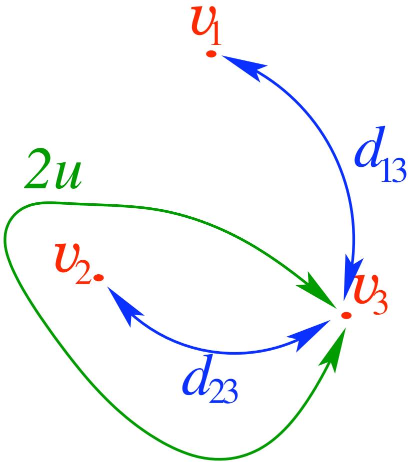
In the scaling limit, we expect all geodesic points to be at distance $o(n^{1/4})$. By this argument, Miermont was able to prove that the unicity of the geodesic between two generic points in the scaling limit of quadrangulations.

Outline

Geodesic loops

Consider a triply pointed quadrangulation (1,2,3) and study the length of the shortest cycle going through 3 separating 1 from 2.



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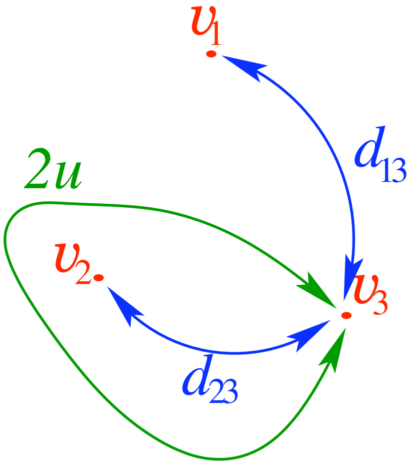
$u \leq \min(d_{13}, d_{23})$



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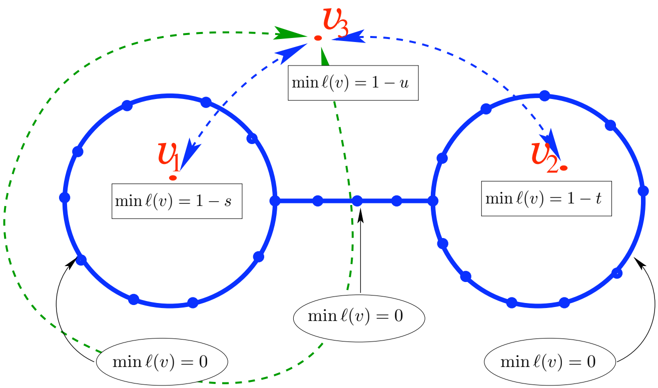

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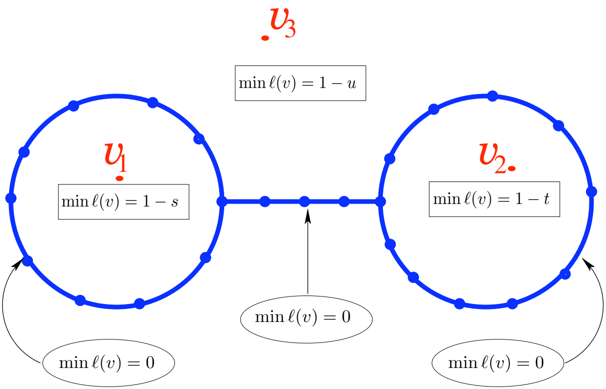

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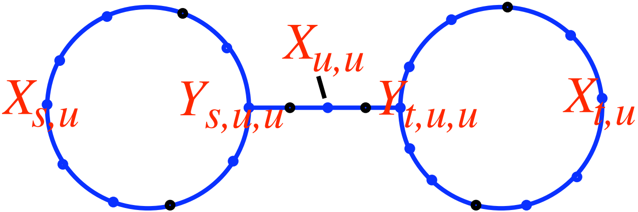

Geodesic loops

We arrive at a generating function:

$$\bar{G}(d_{13}, d_{23}, u) = \Delta_s \Delta_t \Delta_u \; \bar{F}(s, t, u)\Big|_{\substack{s=d_{13}-u \\ t=d_{23}-u}}$$

where

$$\begin{aligned}\bar{F}(s, t, u) &= X_{s,u} X_{t,u} X_{u,u} Y_{s,u,u} Y_{t,u,u} \\ &= \frac{[3][s+1][t+1][u+1]^4[s+2u+3][t+2u+3]}{[1]^3[s+u+1][s+u+3][t+u+1][t+u+3][2u+1][2u+3]}\end{aligned}$$

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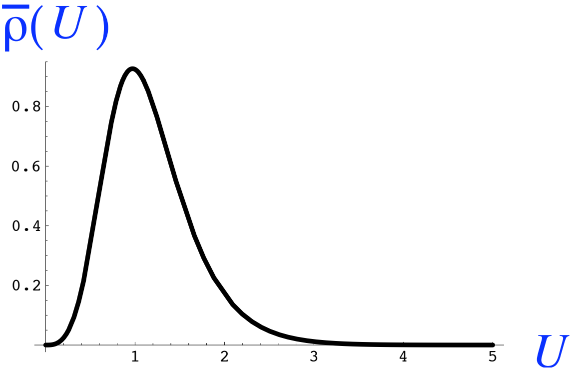
We readily perform the scaling limit and find the law for $U = u \cdot n^{-1/4}$:

$$\rho(\bar{U}) = -\frac{4}{\mathrm{i}\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi \, e^{-\xi^2} \partial_U \left(\frac{\sinh^4(U\sqrt{-3\mathrm{i}\xi/2})}{\sinh^2(2U\sqrt{-3\mathrm{i}\xi/2})} \right)$$

Geodesic loops



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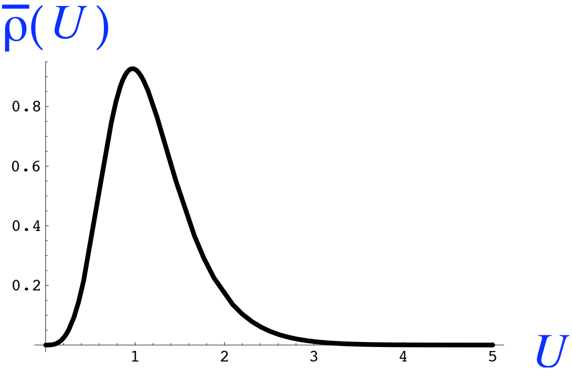

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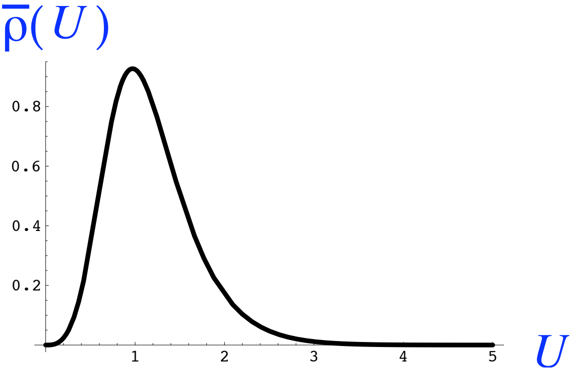

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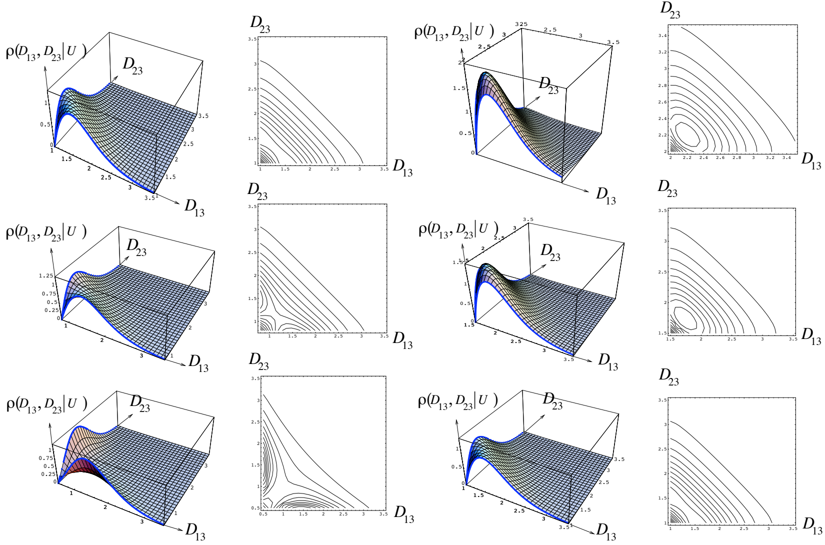
We can also plot:

$$\bar{\rho}(D_{13}, D_{23}|U) = \frac{\bar{\rho}(D_{13}, D_{23}, U)}{\bar{\rho}(U)}.$$

Geodesic loops



$U = 0.5,\ 0.8,\ 1.0,\ 1.5,\ 2.0$

Geodesic loops

Asymptotic regimes:

- $U \ll 1$: one distance is $\propto U$, the other finite.

  $$\bar{\rho}(D_{13}, D_{23}, U) \sim \frac{1}{2}\left(\rho(D_{13})\frac{1}{U}\psi\left(\frac{D_{23}}{U}\right) + \rho(D_{23})\frac{1}{U}\psi\left(\frac{D_{13}}{U}\right)\right)$$

  with

  $$\psi(z) = \frac{3}{2}\cdot\frac{2z-1}{z^4} \qquad z \in [1,\infty)$$

  Consistent with the absence of microscopic cycles separating two macroscopic components.

- $U \gg 1$: both distances are $U + O(U^{-1/3})$

  $$\bar{\rho}(D_{13}, D_{23}, U) \sim (9U)^{2/3}\Phi\left((D_{13}-U)(9U)^{1/3}, (D_{23}-U)(9U)^{1/3}\right)$$

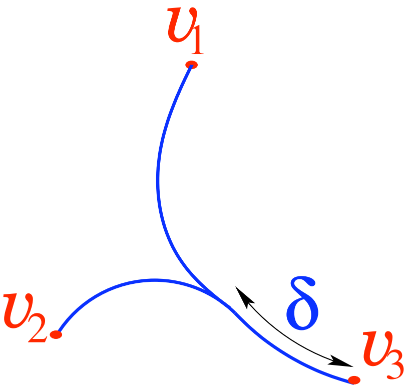
  with

  $$\Phi(z,z') = e^{-(z+z')}\left(2 - e^{-z} - e^{-z}\right).$$

Outline

Confluence of geodesics

Le Gall has shown the surprising phenomenon of *confluence* of geodesics.



Confluence of geodesics

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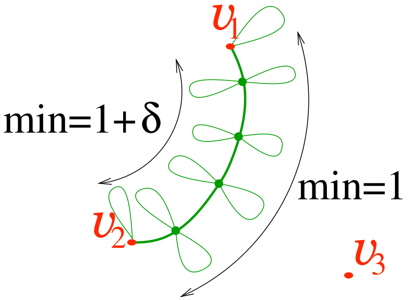

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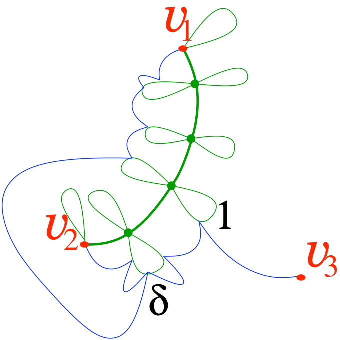

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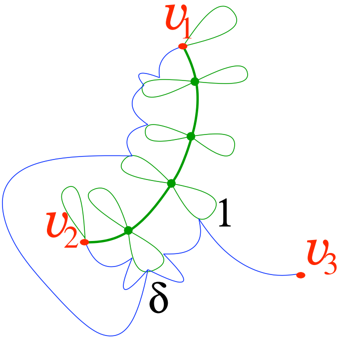
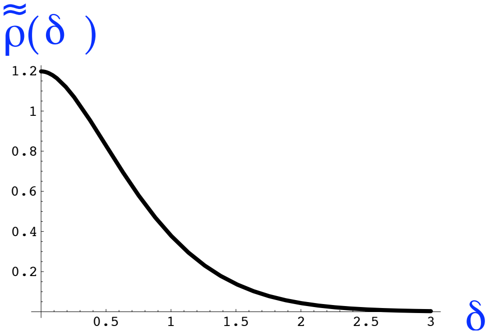


In the discrete setting these correspond to particular geodesics, nevertheless in the scaling limit this makes no difference. We have $\delta \propto n^{1/4}$.

Confluence of geodesics

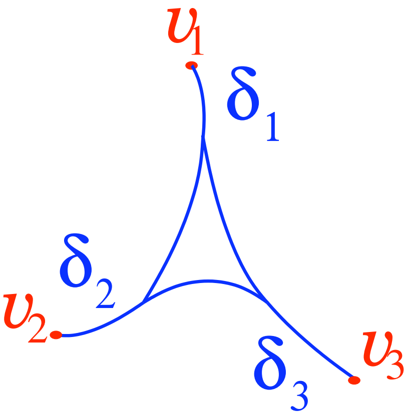
We were able to compute the continuous law for δ ($\delta \to \delta \cdot n^{-1/4}$):

$$\tilde{\tilde{\rho}}(\delta) = \frac{3}{\mathrm{i}\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi \, e^{-\xi^2} \sqrt{-3\mathrm{i}\xi/2}\, e^{-2\delta\sqrt{-3\mathrm{i}\xi/2}}$$



Confluence of geodesics

The shape of a triangle will actually look like:



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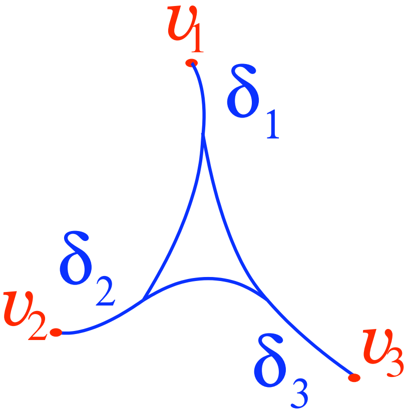
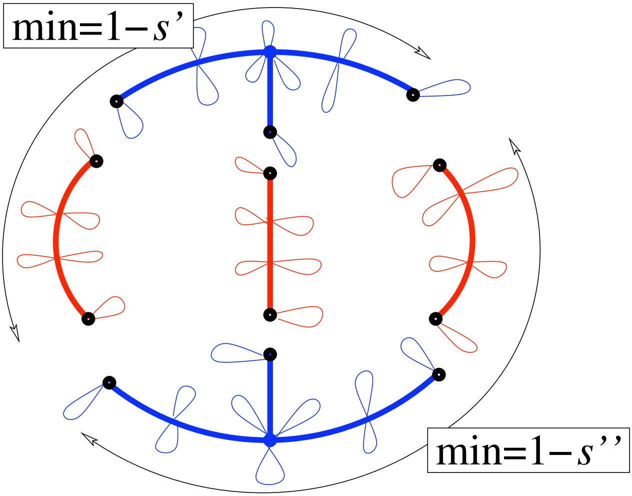


Our computation of the three-point function can be refined into an expression involving six parameters: $d_{12}, d_{23}, d_{23}, \delta_1, \delta_2, \delta_3$.

Confluence of geodesics



$$\max(s', s'') = s = \frac{d_{12} + d_{31} - d_{23}}{2} \qquad |s' - s''| = \delta_1$$

Confluence of geodesics

Similarly we introduce the parameters t', t'', u', u''.
We arrive at a generating function:

$$\Delta_{s'}\Delta_{s''}\Delta_{t'}\Delta_{t''}\Delta_{u'}\Delta_{u''}\left(Y_{s',t',u'}Y_{s'',t'',u''}X_{s',t''}X_{t',u''}X_{u',s''}\right)$$

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Conventions for X become irrelevant in the scaling limit:

$$\partial_{S'}\partial_{S''}\partial_{T'}\partial_{T''}\partial_{U'}\partial_{U''}\frac{3}{\alpha^2}\mathcal{Y}(S',T',U';\alpha)\mathcal{Y}(S'',T'',U'';\alpha)$$

$$\mathcal{Y}(S,T,U;\alpha)=\frac{\sinh(\alpha S)\sinh(\alpha T)\sinh(\alpha U)\sinh(\alpha(S+T+U))}{\sinh(\alpha(S+T))\sinh(\alpha(T+U))\sinh(\alpha(U+S))}$$

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In the canonical ensemble we find a probability density function:

$$\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty}d\xi\,\frac{\xi}{\mathrm{i}}e^{-\xi^2}\left(\cdots\right)\Big|_{\alpha=\sqrt{3\mathrm{i}\xi/2}}$$

Confluence of geodesics

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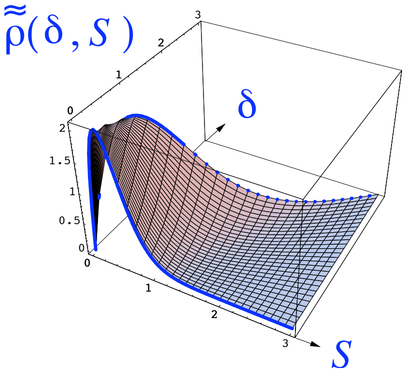
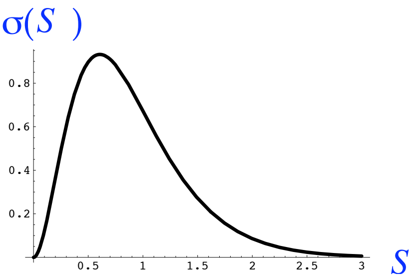
(Grand-canonical) joint law for S and δ_1:

$$\tilde{\tilde{\mathcal{G}}}(S, \delta_1; \alpha) = 6e^{-4\alpha S} e^{2\alpha\delta_1} \qquad S > \delta_1 > 0$$

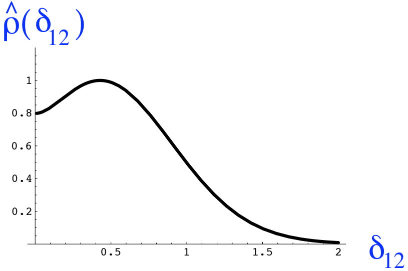




Confluence of geodesics

Side of the "inner" triangle:

$$\delta_{12} = D_{12} - \delta_1 - \delta_2$$



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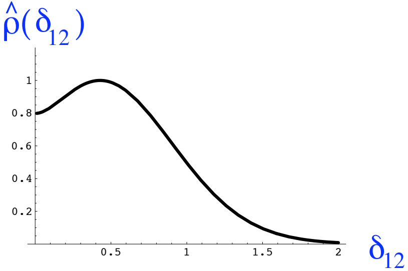

We can also study the area of the inner triangle. We find it has an area βn where $\beta \in [0, 1]$ has density:

$$\frac{\sqrt{\pi}}{\Gamma(1/4)^2} \frac{1}{(\beta(1-\beta))^{3/4}}$$

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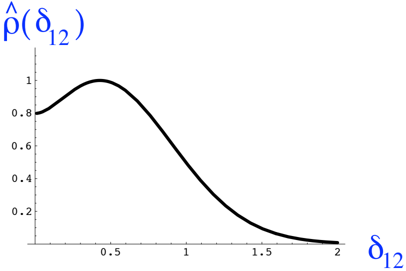

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(same as the area within a geodesic loop)

Conclusion

We have computed a number of properties of geodesics in planar quadrangulations, both in the local and scaling limit.

- the mean number of geodesics between two given points at distance $i \gg 1$ is $3 \cdot 2^i$
- the mean number of geodesic points at a given generic position is 3
- geodesic loops and confluence of geodesics can be quantitatively studied.

Still, the structure of a large random quadrangulation remains mysterious, inbetween tree and sphere.