On Entire Solutions of Linear Difference Equations with Polynomial Coefficients

S. A. Abramov

Computing Centre

of Russian Academy of Sciences

We will deal with solutions of linear difference equations with polynomial coefficients:

$$a_d(z)y(z+d)+\cdots+a_1(z)y(z+1)+a_0(z)y(z)=0, \qquad (1)$$

$a_1(z), a_2(z), \ldots, a_{d-1}(z) \in \mathbb{C}[z], \quad a_0(z), a_d(z) \in \mathbb{C}[z] \setminus \{0\}$, $\gcd(a_0(z), \ldots, a_d(z)) = 1$.

We can associate with equation (1) the linear difference operator

$$L = a_d(z)E^d + \cdots + a_1(z)E + a_0(z), \qquad (2)$$

where E is the shift operator: $E(y(z)) = y(z+1)$. Equation (1) can be rewritten in the form $L(y) = 0$.

For short, we will say about solutions of L instead of solutions of the equation $L(y) = 0$.

It is known that any operator of the form (2) has a fundamental system of entire solutions (Praagman, 1986). We strengthen this result:

Theorem 1 *The $\mathbb{C}$-linear space of sequences which are restrictions to $\mathbb{Z}$ of entire solutions of operator (2) has dimension d. A basis for this space can be found algorithmically.*

Example 1 *The* Γ*-function* $\Gamma(z)$ *is a meromorphic solution of* $L = E - z$. *This function has finite values when* $z = 1, 2, \ldots,$ *and simple poles when* $z = 0, -1, -2, \ldots$

Set

$$\sigma(z) = \frac{\sin 2\pi z}{2\pi}.$$

If we multiply $\Gamma(z)$ *by* $\sigma(z)$, *then we get an entire solution of* L *(since* $\sigma(z) = \sigma(z+1)$*). The sequence*

$$c_n = \begin{cases} 0, & \text{if } n > 0, \\ \frac{(-1)^{-n+1}}{(-n+1)!}, & \text{if } n \leq 0. \end{cases} \tag{3}$$

is the restriction of this solution to $\mathbb{Z}$. *Notice that if* $n \leq 0$ *then* $c_n = \mathrm{Res}_{z=n}\Gamma(z)$ *since*

$$\sigma(z-n) = z - n + o(z-n), \;\; z \to n, \;\; n \in \mathbb{Z}.$$

This example demonstrates a simple trick which can help to construct the solutions we need.

We add the following. Let $\varphi(z)$ be a meromorphic solution of an operator L of the form (2) and $u \in \mathbb{C}$. Then the orders of all poles of φ that belong to $u + \mathbb{Z}$ are bounded. If N is the maximum of such orders, then we can consider the solution $(\sigma(z-u))^N \varphi(z)$. This solution has no pole in the set $u + \mathbb{Z}$ and its restriction to $u + \mathbb{Z}$ is a non-zero sequence.

Let's go back to entire solutions. The following statement can be proven quite easy

Let $\varphi_1(z), \varphi_2(z), \ldots, \varphi_{d+1}(z)$ be entire solutions of an operator L of order d, then the sequences

$$(\varphi_1(n)),\ \ (\varphi_2(n)),\ \ \ldots\ \ ,(\varphi_d(n)),\ \ (\varphi_{d+1}(n)), \quad n \in \mathbb{Z},$$

are $\mathbb{C}$-linearly dependent.

The question is: How to prove that there exist the entire solutions $\varphi_1(z), \varphi_2(z), \ldots, \varphi_d(z)$ such that the sequences

$$(\varphi_1(n)),\ \ (\varphi_2(n)),\ \ \ldots\ \ ,(\varphi_d(n)), \quad n \in \mathbb{Z},$$

are $\mathbb{C}$-linearly independent?

The following theorem helps very much.

Theorem 2 *(Ramis-Barkatou) An operator* $L \in \mathbb{C}[z, E]$ *of order* d *has meromorphic solutions* $\psi_1(z), \psi_2(z), \ldots, \psi_d(z)$ *such that for some integer* l

(a) $\psi_i(l + j) = \delta_{i,j}, \;\; i, j = 1, 2, \ldots, d;$

(b) $\psi_i(z)$ *has no poles in the half-plane* $\mathrm{Re}\, z > l, \;\; i = 1, 2, \ldots, d.$

Notice that all possible poles of $\psi_1(z), \psi_2(z), \ldots, \psi_d(z)$ belongs to a set of the form

$$\bigcup_{j=1}^{k}(u_j - \mathbb{N}), \quad u_1, u_2, \ldots, u_k \in \mathbb{C}, \tag{4}$$

since the coefficients of L are polynomials and due to (b).

At the first glance we can prove easy what we need using the statement of this theorem together with the simple trick that was demonstrated in Example 1. But this is not correct.

Example 2 *For any* $d \geq 1$ *we can construct* $L \in \mathbb{C}[z, E]$, $\operatorname{ord} L = d$, *having rational solutions*

$$\frac{1}{z}, \quad \frac{1}{z^2}, \quad \dots \quad , \frac{1}{z^d}. \tag{5}$$

Set again

$$\sigma(z) = \frac{\sin 2\pi z}{2\pi}.$$

If we multiply solutions (5) by, resp., $\sigma(z), (\sigma(z))^2, \dots, (\sigma(z))^d$, *then the restriction to* $\mathbb{Z}$ *of any product will be equal to the sequence* $(\delta_{n,0})$. *The first impression is such that the restriction to* $\mathbb{Z}$ *of any meromorphic solution of* L *that has no poles in* $\mathbb{Z}$ *is a sequence of the form*

$$c_n = \begin{cases} u, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}$$

$n \in \mathbb{Z}$, $u \in \mathbb{C}$.

But we can see that L has besides (5), e.g., the entire solution

$$\frac{1}{z} - \sigma(z)\frac{1}{z^2}$$

whose restriction to $\mathbb{Z}$ *is the sequence h:*

$$h_n = \begin{cases} 0, & \text{if } n = 0, \\ \frac{1}{n}, & \text{if } n \neq 0, \end{cases}$$

$n \in \mathbb{Z}$.

It is easy to show that L has d meromorphic (even entire) solutions whose restrictions to $\mathbb{Z}$ are $\mathbb{C}$-linearly independent sequences $h^{(1)}, h^{(2)}, \ldots, h^{(d)}$:

$$h_n^{(j)} = \begin{cases} 0, & \text{if } n = 0, \\ \frac{1}{n^j}, & \text{if } n \neq 0, \end{cases}$$

$j = 1, 2, \ldots, d-1$,

$$h_n^{(d)} = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

As for the general case, we can prove the following proposition.

Proposition 1 *Let $\psi_1(z), \psi_2(z), \ldots, \psi_d(z)$ be as in Theorem 2. Then the set*

$$\mathbb{C}\left[\sigma(z), (\sigma(z))^{-1}\right]\psi_1(z) + \cdots + \mathbb{C}\left[\sigma(z), (\sigma(z))^{-1}\right]\psi_d(z) \tag{6}$$

contains $\chi_1(z), \chi_2(z), \ldots, \chi_d(z)$ such that

- *the poles of $\chi_1(z), \chi_2(z), \ldots, \chi_d(z)$ belong to a set of the form*

  $$\bigcup_{j=1}^{m}(w_j + \mathbb{Z}), \tag{7}$$

  where the complex numbers w_j, $j = 1, 2, \ldots, m$, are not integer;

- *the sequences*

  $$(\chi_1(n)), \ (\chi_2(n)), \ \ldots \ , (\chi_d(n)), \quad n \in \mathbb{Z},$$

  are $\mathbb{C}$-linearly independent.

(Of course, any function from the set (6) is a meromorphic solution of the original L.)

This proposition gives an opportunity to prove that L of the form (2) has the entire solutions $\varphi_1(z), \varphi_2(z), \dots, \varphi_d(z)$ such that the sequences $(\varphi_1(n))$, $(\varphi_2(n))$, $\dots$ $, (\varphi_d(n))$ are $\mathbb{C}$-linearly independent.

Consider the mentioned above meromorphic solutions $\chi_1(z), \chi_2(z), \ldots, \chi_d(z)$ of L. Let N be the maximal value of the orders of the poles of $\chi_1(z), \chi_2(z), \ldots, \chi_d(z)$. Let

$$\lambda(z) = \prod_{j=1}^{m} \sin^{2N} \pi(z - w_j)$$

(see formula (7)). The entire functions

$$\varphi_1(z) = \lambda(z)\chi_1(z), \;\; \varphi_2(z) = \lambda(z)\chi_2(z), \;\; \ldots \;\; , \varphi_d(z) = \lambda(z)\chi_d(z)$$

are solutions of L since $\sin^2 \pi u = \sin^2 \pi(u+1)$ for any $u \in \mathbb{C}$.

The restrictions of these entire solutions to $\mathbb{Z}$ are $\mathbb{C}$-linearly independent, since up to the non-zero factor

$$\prod_{j=1}^{m} \sin^{2N} \pi w_j$$

they are equal to the restrictions of the functions $\chi_1(z), \chi_2(z), \ldots, \chi_d(z)$ to $\mathbb{Z}$.

Example 3 *The meromorphic (rational) function*

$$\psi(z) = \frac{1}{z(2z+1)(3z+1)}$$

is a solution of the operator
$L = (z+1)(2z+3)(3z+4)E - z(2z+1)(3z+1)$.

The multiplication of $\psi(z)$ *by*

$$\sigma(z)\sin^2\pi\left(z+\frac{1}{3}\right)$$

gives an entire solution $\varphi(z)$ *of* L *which vanishes at* $(-\frac{1}{2}+\mathbb{Z})\cup(-\frac{1}{3}+\mathbb{Z})$ *and* $(\varphi(n)) = (\delta_{n,0})$ *(notice that the factor* $\sigma(z)$ *eliminates the poles* 0 *and* $-\frac{1}{2}$ *of* $\varphi(z)$*).*

The restrictions to $\mathbb{Z}$ of a meromorphic solution of L which has no poles in $\mathbb{Z}$ will be called a *submeromorphic* (sequential) solution of L. The $\mathbb{C}$-linear space of submeromorphic solutions of L will be denoted by $V_{\mathrm{sm}}(L)$.

A *subanalytic* solution of L is a restriction to $\mathbb{Z}$ of a single-valued analytic solution $\varphi(z)$ such that $\mathbb{Z} \subset \mathrm{dom}(\varphi)$. The $\mathbb{C}$-linear space of subanalytic solutions of L will be denoted by $V_{\mathrm{sa}}(L)$.

Finally, let us denote the $\mathbb{C}$-linear space of restrictions to $\mathbb{Z}$ of entire solutions of L by $V_{\mathrm{se}}(L)$.

The following theorem can be proven:

Theorem 3 *The equalities*

$$V_{\mathrm{se}}(L) = V_{\mathrm{sa}}(L) = V_{\mathrm{sm}}(L)$$

hold.

We will also consider the space $V_{\mathrm{sf}}\ (L)$ of so-called subformal sequential solutions. This will help us to prove that a basis for the space $V_{\mathrm{se}}(L)$ can be found algorithmically.

As usual, we denote by $\mathbb{C}[[\varepsilon]]$ the ring of formal Taylor power series in ε and by $\mathbb{C}((\varepsilon))$ the field of formal Laurent series, i.e. the quotient field of the ring $\mathbb{C}[[\varepsilon]]$ (here ε is a new variable, rather than a "small number"). Any sequence $F : \mathbb{Z} \to \mathbb{C}((\varepsilon))$ will be called a *formal* sequence.

If $a(z)$ is a polynomial or a rational function then we set $\hat{a}(z) = a(z + \varepsilon)$, here ε is a variable. We associate with L the operator

$$\hat{L} = \hat{a}_d(z)E^d + \cdots + \hat{a}_1 E(z) + \hat{a}_0(z) = a_d(z+\varepsilon)E^d + \cdots + a_0(z+\varepsilon)$$

which acts on formal sequences.

The operator $\hat{L}$ is called the *deformation* of L (M. van Hoeij).

We will call a *formal sequential* solution of $\hat{L}$ a solutions of the form $F : \mathbb{Z} \to K((\varepsilon))$.

Let $\psi : \mathbb{C} \to \mathbb{C}$ be a meromorphic function. For each $n \in \mathbb{Z}$ expand

$$\psi(z) = c_{n,\rho_n}(z-n)^{\rho_n} + c_{n,\rho_n+1}(z-n)^{\rho_n+1} + \dots$$

with $\rho_n \in \mathbb{Z}$ and $c_{n,\rho_n} \neq 0$. Define the formal sequence

$$\hat{\psi} : \mathbb{Z} \to \mathbb{C}((\varepsilon)), \quad \hat{\psi} = (\hat{\psi}_n),$$

setting

$$\hat{\psi}_n = c_{n,\rho_n}\varepsilon^{\rho_n} + c_{n,\rho_n+1}\varepsilon^{\rho_n+1} + \dots,$$

$n \in \mathbb{Z}$.

It is possible to prove that

$$L(\psi) = 0 \implies \hat{L}(\hat{\psi}) = 0. \tag{8}$$

A formal solution $F : \mathbb{Z} \to \mathbb{C}[[\varepsilon]]$ of $\hat{L}$ will be called a *Taylor* formal solution. A sequential solution f will be called *subformal* (sequential) solution of L if $\hat{L}$ has a formal Taylor solution F, such that f_n is the constant term of the series F_n, $n \in \mathbb{Z}$. The set $V_{\mathrm{sf}}\ (L)$ of subformal solutions of L is evidently a $\mathbb{C}$-linear space.

The following theorem can be proven.

Theorem 4 $\dim V_{\mathrm{sf}}\ (L) = \operatorname{ord} L$.

Theorems 3, 4 and (8) imply that

$$V_{\mathrm{sf}}\ (L) = V_{\mathrm{se}}(L) = V_{\mathrm{sa}}(L) = V_{\mathrm{sm}}(L).$$

Therefore it is sufficient to construct a basis of $V_{\mathrm{sf}}\ (L)$ to get a basis of, e.g., $V_{\mathrm{se}}(L)$.

A segment of integer numbers

$$I = \{k, k+1, \ldots, l\}, \;\; k, l \in \mathbb{Z}, \;\; k \leq l,$$

is an *essential segment* of (2) if

- the polynomial $a_d(z-d)$ has no integer roots $> l$,
- the polynomial $a_0(z)$ has no integer roots $< k$,
- $\#(I) \geq d$.

If I is an essential segment of operator (2) then any sequential solution c is uniquely determined by the values c_n, $n \in I$. Therefore if we want to describe $V_{\mathrm{sf}}\,(L)$, then it is sufficient to find a basis of the restriction of $V_{\mathrm{sf}}\,(L)$ to I.

The algorithm is based on the algorithm by Abramov&van Hoeij (2003) for finding values of subformal solutions, the idea of that algorithm is as follows.

Let $q \in \mathbb{Z}$ and

$$F_q, F_{q+1}, \ldots, F_{q+d-1}$$

be given elements of $K[[\varepsilon]]$, then, theoretically speaking, by using the operator $\hat{L}$, we can compute any element F_p of the sequential solution $F = (F_n)$ of the equation $\hat{L}(y) = 0$.

It may be that $F_p \in K((\varepsilon)) \setminus K[[\varepsilon]]$ for a given integer $p \notin \{q, q+1, \ldots, q+d-1\}$. Starting with q, p we can write down in advance a finite set $C_{q,p}$ of linear equations for a finite number of coefficients of power series $F_q, F_{q+1}, \ldots, F_{q+d-1}$ which guarantee that $F_p \in K[[\varepsilon]]$.

Indeed, set

$$\begin{array}{ccccccccc}
F_q & = & u_{q,0} & + & u_{q,1}\varepsilon & + & u_{q,2}\varepsilon^2 & + & \cdots, \\
F_{q+1} & = & u_{q+1,0} & + & u_{q+1,1}\varepsilon & + & u_{q+1,2}\varepsilon^2 & + & \cdots, \\
\vdots & & \vdots & & \vdots & & \vdots & & \\
F_{q+d-1} & = & u_{q+d-1,0} & + & u_{q+d-1,1}\varepsilon & + & u_{q+d-1,2}\varepsilon^2 & + & \cdots,
\end{array} \tag{9}$$

where series on the right are generic. When we compute F_p we get a series, and each of its coefficients is a linear form in a finite set of $u_{i,j}$. The series F_p may contain negative exponents of ε. We can find desired conditions on the coefficients $u_{i,j}$ in (9) after equating the corresponding coefficients to zero.

If $q \in \mathbb{Z}$ is fixed then the systems $C_{q,p}$ for any integer $p > q + d - 1$ can be found algorithmically using truncated series (taking into account the terms of power series (9) till ε^m where m is the sum of multiplicities of all integer roots of the polynomial $a_d(z - d)$). It is also possible to find the linear form $l_{q,p}$ which represents the coefficient of ε^0 of the series F_p.

Now we are able to describe how to construct a basis of the restrictions to I of all subformal solutions of L.

Let $I = \{k, k+1, \ldots, l\}$ be an essential segment of L. If $l = k + d - 1$ then we can take any d $\mathbb{C}$-linearly independent elements of $\mathbb{C}^d$ and this will be a basis of the restriction of $V_{\mathrm{sf}}\ (L)$ to I. Suppose that $l > k + d - 1$. Take $q = k$ and construct $C_{q,p}$, $l_{q,p}$ for $p = k + d, k + d + 1, \ldots, l$. Add to linear equations from all constructed $C_{q,p}$ the equations

$$u_{p,0} = l_{q,p}, \quad p = k + d, k + d + 1, \ldots, l.$$

Denote by $\mathcal{A}$ the obtained system of linear algebraic equations.

Let us construct a basis of the solution space of $\mathcal{A}$ and then construct the projection of each vector of this basis into the space of vectors $(u_{k,0}, u_{k+1,0}, \ldots, u_{l,0})$. Taking any basis of the $\mathbb{C}$-linear space generated by such projections we get a basis of the restrictions to I of all subformal solutions of the operator L.

Example 4 *Let*

$$L = z^2E^2 + (1+z^2)E - z$$

and F be a formal solution of $\hat{L}$. The segment $I = \{0,1,2\}$ is an essential segment of L.

Write

$$F_0 = u_{0,0} + u_{0,1}\varepsilon + u_{0,2}\varepsilon^2 + O(\varepsilon^3),$$

$$F_1 = u_{1,0} + u_{1,1}\varepsilon + u_{1,2}\varepsilon^2 + O(\varepsilon^3).$$

We calculate using $\hat{L}$:

$$F_2 = -\frac{u_{1,0}}{\varepsilon^2} + \frac{-u_{1,1} + u_{0,0}}{\varepsilon} - u_{1,0} - u_{1,2} + u_{0,1} + O(\varepsilon).$$

We find $C_{0,2} = \{-u_{1,0} = 0,\ u_{0,0} - u_{1,1} = 0\}$ and $l_{0,2} = -u_{1,0} - u_{1,2} + u_{0,1}$.

We get the system $\mathcal{A}$:

$$\begin{array}{rrrrrrrl} & -u_{1,0} & & & & & = & 0, \\ u_{0,0} & & & & -u_{1,1} & & = & 0, \\ & u_{1,0} & +u_{2,0} & -u_{0,1} & & +u_{1,2} & = & 0. \end{array}$$

A basis of the space of its solutions

$$(u_{0,0},\ u_{1,0},\ u_{2,0},\ u_{0,1},\ u_{1,1},\ u_{1,2})$$

is

$$(0,0,1,1,0,0),\ (0,0,-1,0,1,0),\ (1,0,0,0,0,1).$$

The projections of these vectors into the space of vectors $(u_{0,0}, u_{1,0}, u_{2,0})$ *are*

$$(0,0,1),\ (0,0,-1),\ (1,0,0),$$

and a basis of the space generated by these three vectors is

$$(0,0,1),\ (1,0,0). \tag{10}$$

It follows that the vectors (10) give a basis of subformal solutions restricted to I.

Subanalytic solutions have applications in computer algebra. It can be shown that some implementations of certain well-known summation algorithms (Gosper, Zeilberger, Accurate Summation) in existing computer algebra systems work correctly when the input sequence is a subanalytic solution of an equation or a system, but can give incorrect results for some sequential solutions.

Example 5 *Let*

$$L = 2(z+1)(z-2)E - (2z-1)(z-1). \tag{11}$$

There are two sequential solutions

$$c_n^{(1)} = \frac{\Gamma(2n-2)}{\Gamma(n+1)\Gamma(n-2)4^n}, \quad n \in \mathbb{Z}$$

(the limit

$$\lim_{z \to z_0} \frac{\Gamma(2z-2)}{\Gamma(z+1)\Gamma(z-2)4^z}$$

exists and is finite even when $\Gamma(2z-2)$ *has a pole at* z_0*), and*

$$c_n^{(2)} = \frac{\binom{2n-3}{n}}{4^n}, \quad n \in \mathbb{Z}.$$

The sequences $c^{(1)}$ *and* $c^{(2)}$ *coincide when* $n > 1$ *or* $n < 0$*, but in combinatorics* $\binom{2n-3}{n}$ *is usually defined to be* -1 *when* $n = 1$ *and* 1 *when* $n = 0$*, while* $\frac{\Gamma(2n-2)}{\Gamma(n+1)\Gamma(n-2)}$ *is equal to* $-\frac{1}{2}$ *when* $n = 1$*, and is equal to* $\frac{1}{2}$ *when* $n = 0$*.*

Gosper's algorithm succeeds on L producing

$$R(z) = \frac{2z(z+1)}{z-2}.$$

At first glance this implies that the DNLF

$$\sum_{n=0}^{w-1} c_n \;= R(w)c_w - R(0)c_0 \tag{12}$$

is correct for any sequential solution c of L.

But this formula is not in general correct when, e.g., $c = c^{(2)}$*. Formula (12) would give*

$$\sum_{n=0}^{w-1} \frac{\binom{2n-3}{n}}{4^n} = \frac{2w(w+1)\binom{2w-3}{w}}{(w-2)4^w}$$

what is true only if $w = 1$*.*

The fact is that we are not able to define the element b_2 *of the sequence* $b_n = R(n)c_n^{(2)}$ *in such a way that* $b_{n+1} - b_n = c_n^{(2)}$ *for all* $n \in \mathbb{Z}$.

This gives rise to defects in many implementations of summation algorithms.

At the same time formula (12) is correct for any $w \geq 1$ when $c = c^{(1)}$:

$$\sum_{n=0}^{w-1} \frac{\Gamma(2n-2)}{\Gamma(n+1)\Gamma(n-2)4^n} = \frac{2w(w+1)\Gamma(2w-2)}{(w-2)\Gamma(w+1)\Gamma(w-2)4^w},$$

and this is due to the solution $c^{(1)}$ is a subanalytic solution of L (and even more: $c^{(1)}$ is the restriction of an entire solution of L to $\mathbb{Z}$).

Subanalytic solutions of L are safe for applying summation algorithms (this can be proven), but the condition of the subanalyticy is not a necessary condition for correct applicability of the summing algorithms: there exist examples where the dimension of the space of "nice" sequential solutions is $> d$.

Example 6 *If $L = zE - (z+1)$, then Gosper's algorithm produces the one parametric family of summing operators (rational functions)*

$$\frac{z-1}{2} + \frac{\alpha}{z}, \quad \alpha \in \mathbb{C}.$$

If we take $\alpha = 0$ we get $R = \frac{z-1}{2}$. Then any sequential solution of L can be multiplied by R.

The dimension of the space of all sequential solutions of L is 2, a basis is

$$c_n^{(1)} = n, \quad c_n^{(2)} = |n|, \quad n \in \mathbb{Z}.$$

The sequential solution $c^{(1)}$ *is subanalytic since L has the analytic solution* $y(z) = z$*. But sequential solution* $c^{(2)}$ *is not subanalytic since the dimension of the space of subanalytic solutions of a first order operator is* 1*.*

The results presented in this talk were obtained by the author jointly with M. Barkatou, M. van Hoeij, M. Petkovšek.