Random sampling of plane partitions

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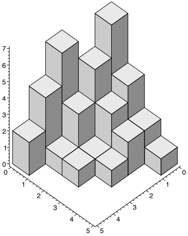
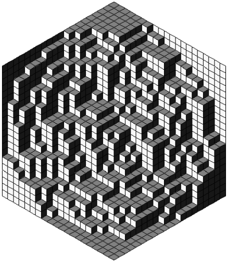
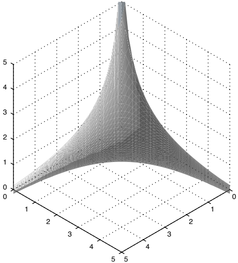




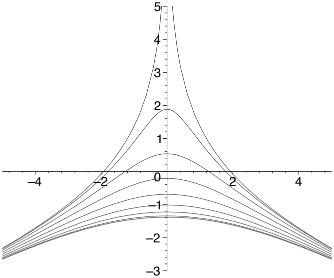


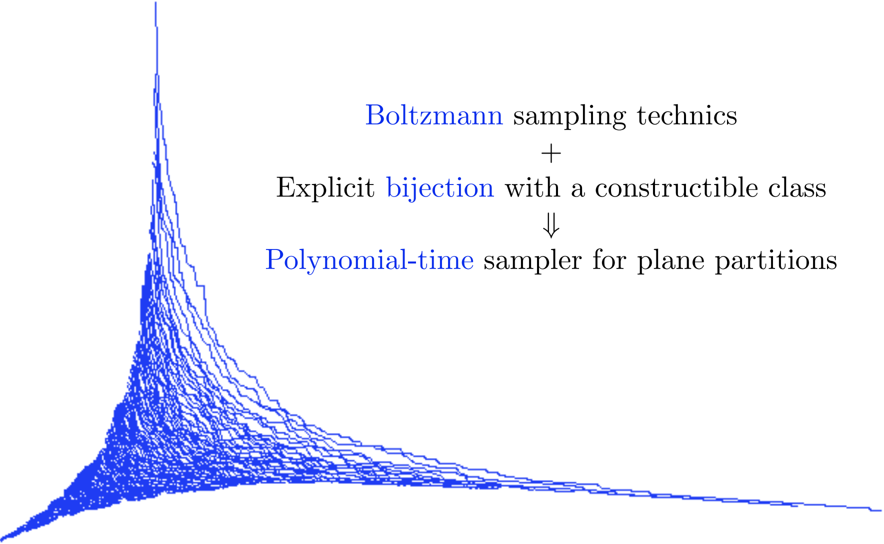
Context

- Young tableaux : natural generalization of integer partitions in 3D,
- huge literature, e.g. the Alternating Sign Matrix Conjecture (Zeilberger 1995),
- Mac Mahon : beautiful (and simple) generating function ($\sim$ 1912)
- for long, no bijective proof,
- Krattenthaler, 1999, proof based on interpretation the hook-length formula,
- sampling of plane partitions in a box $a \times b \times c$:
  $\rightarrow$ hexagon tilings by rhombi,
- 2002 : Pak's bijection for general planes partitions,
- 2004 : Boltzmann sampling
- today : efficient samplers for some classes of plane partitions.

Motivations

- mathematics,
- statistical physics,
- random sampling according to a natural parameter (volume),
- very large object → observation of limit properties,
- in particular : limit shape
  - Cerf and Kenyon,
  - Okounkov and Reshetikhin
- phenomena such as frozen boundaries,
- ...





Boltzmann sampling technics

+

Explicit bijection with a constructible class

$\Downarrow$

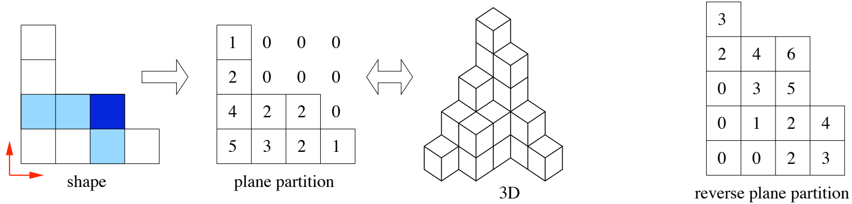
Polynomial-time sampler for plane partitions

Plan of the talk

1. Pak's bijection
2. Boltzmann sampler
3. Analysis of Complexity

Planes partitions

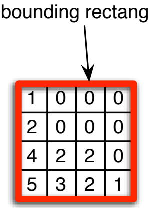
- λ : Integer partition $\simeq$ Shape of plane partition
  e.g. : $\lambda = \{4, 3, 1, 1\}$.
- $h(i, j)$: hook length of the cell (i, j)
- Plane partitions of shape λ ($\mathcal{P}$)
  - λ filled with integers > 0, decreasing in both dimensions
  - matrix filled with integers ≥ 0, decreasing in both dimensions
- Reverse plane partition of shape λ ($\mathcal{RP}$)
  λ filled with integers ≥ 0 that are increasing in both dimensions
- Size of a plane partition : sum of the entries

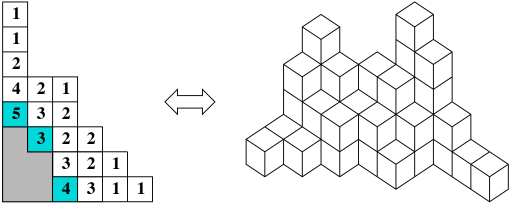


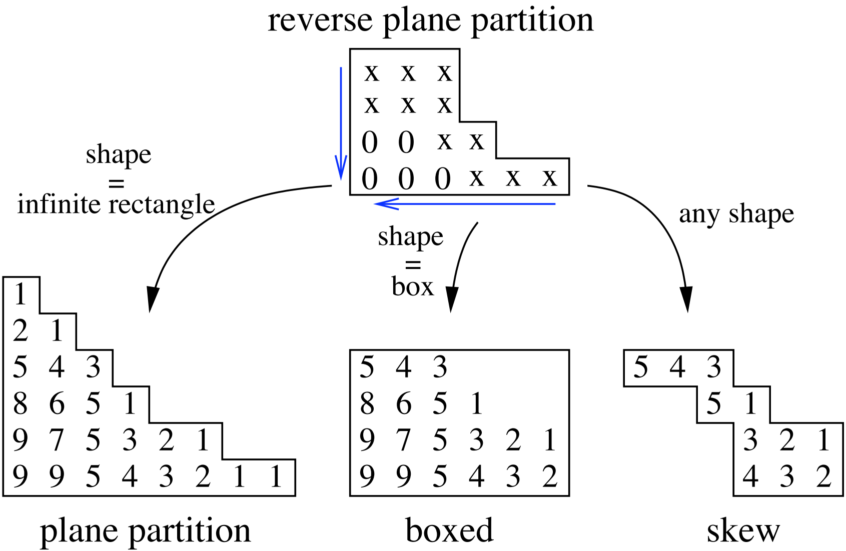
Boxed and skew planes partitions

- Bounding rectangle of a plane partition
  the smallest rectangle containing all the non-zero cells
- $(a \times b)$-boxed plane partitions ($\mathcal{P}_{a,b}$)
  the size of the bounding rectangle is at most $(a \times b)$
- Skew plane partitions ($\mathcal{S}$)
  plane partition of shape λ/μ, where λ, μ are integer partitions and $\lambda \supset \mu$
- Corner of a skew plane partition

$$\mathcal{S} \equiv \mathcal{RP}$$






reverse plane partition
x x x
x x x
0 0 x x
0 0 0 x x x
shape = infinite rectangle
shape = box
any shape
1
2 1
5 4 3
8 6 5 1
9 7 5 3 2 1
9 9 5 4 3 2 1 1
plane partition
5 4 3
8 6 5 1
9 7 5 3 2 1
9 9 5 4 3 2
boxed
5 4 3
5 1
3 2 1
4 3 2
skew

Counting plane partitions

Hook content formula :

$$\sum_{A \in \mathcal{RP}(\lambda)} z^{|A|} = \prod_{(i,j)\in[\lambda]} \frac{1}{1 - z^{h(i,j)}}$$

Set λ to be an infinite rectangle :

$$\prod_{i,j\geq 0} \frac{1}{1 - z^{i+j+1}}$$

Generating function of plane partitions (Mac Mahon, 1912) :

$$P(z) = \prod_{r\geq 1} (1 - z^r)^{-r}$$

- combinatorial isomorphisms with constructible classes (symbolic methods)

$$\mathcal{P} \simeq \mathcal{M} \ , \quad \mathcal{P}_{a,b} \simeq \mathcal{M}_{a,b} \quad \text{and} \quad \mathcal{S}_D \simeq \mathcal{M}_D$$

- non-trivial bijection, for long, non constructive proof...

Isomorphic classes

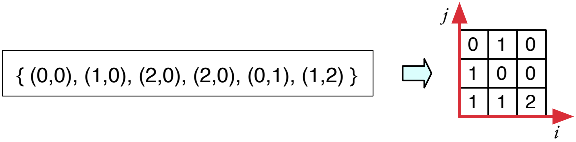
$$\prod_{i,j\geq 0} \frac{1}{1-z^{i+j+1}} = \prod_{i,j\geq 0} \textsc{Seq}(\mathcal{Z} \times \mathcal{Z}^i \times \mathcal{Z}^j) = \textsc{MSet}(\mathcal{Z} \times \textsc{Seq}(\mathcal{Z})^2)$$

- $\mathcal{M} = \textsc{MSet}(\mathbb{N}^2)$ $\sim$ multiset of pairs of integers

  $\rightarrow$ example : $\{(0,0),(1,0),(2,0),(2,0),(0,1),(1,2)\}$, size $= 15$

  $\rightarrow$ size of (i,j) : $(i+j+1)$

- Diagram of an element $\in \mathcal{M}$

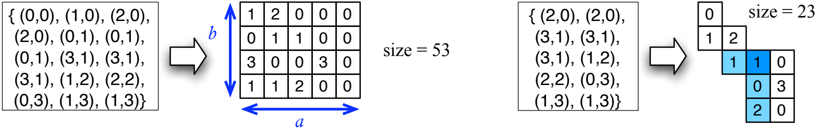


size $=$ 15

$|D| = \sum_{i,j} m_{i,j}(i+j+1)$

$\rightarrow$ sum of the hook lengths weighted by the values of the cells.

- $$\mathcal{M}_{a,b} = \text{MSET}(\mathcal{Z} \times \text{SEQ}_{<a}(\mathcal{Z}) \times \text{SEQ}_{<b}(\mathcal{Z}))$$
  $$= \prod_{\substack{0 \le i < a \\ 0 \le j < b}} \text{SEQ}(\mathcal{Z} \times \mathcal{Z}^i \times \mathcal{Z}^j)$$
  $$\sim \text{MSET}(\mathbb{N}_{<a} \times \mathbb{N}_{<b})$$
- $$\mathcal{M}_D = \prod_{(i,j)\in D} \text{SEQ}(\mathcal{Z} \times \mathcal{Z}^{i-\ell(i)} \times \mathcal{Z}^{j-d(j)}) = \prod_{(i,j)\in D} \text{SEQ}(\mathcal{Z}^{h(i,j)})$$
- Diagrams

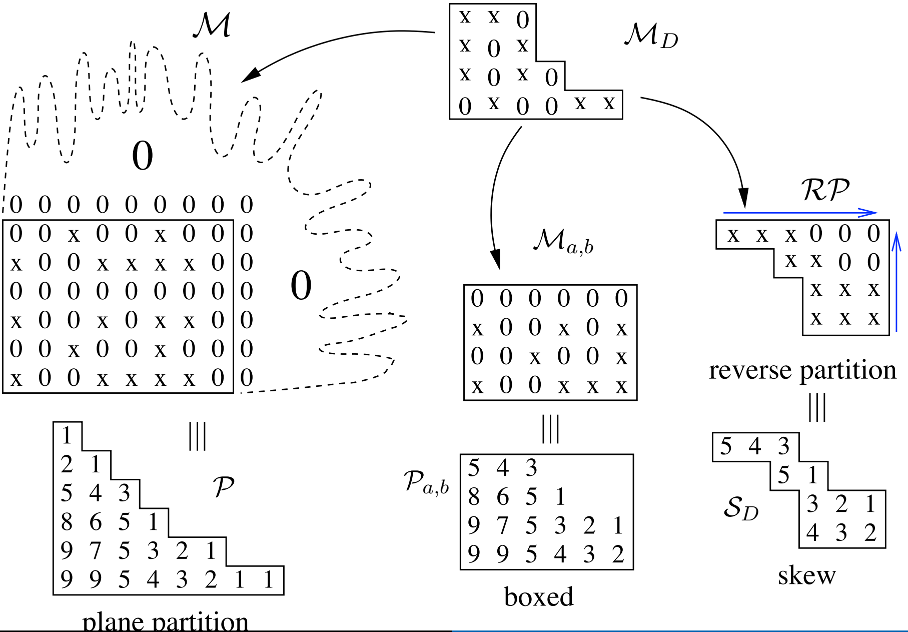


- Hook length of $(i, j) \in D : h(i, j) = (i - \ell(i)) + (j - d(j)) + 1$

  $\ell(i) \leftarrow$ min. abscissa such that $(\ell(i), j) \in D$

  $d(j) \leftarrow$ min. ordinate such that $(i, d(j)) \in D$

Summary


𝓜
𝓜_D
𝓡𝓟
𝓜_a,b
reverse partition
𝓟
𝓟_a,b
𝓢_D
boxed
skew
plane partition

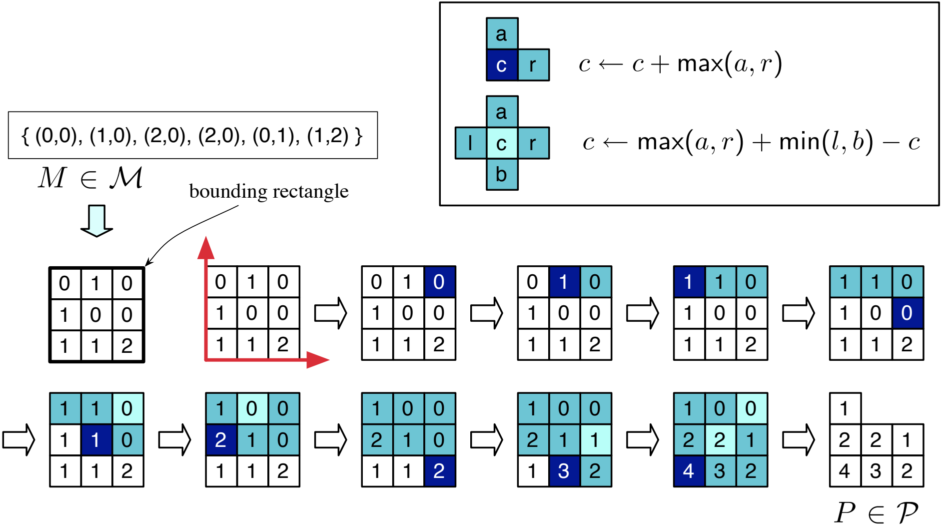
Pak's bijection

Pak's bijection – principles

- sequential update of the corners of the multiset M
- at each step, the current plane partition (of shape λ) correspond to the restriction of M to λ
- prop. 1 : for any corner, the value of the cell, in the plane partition = the maximum value of a monotone path, in the multiset.
- prop. 2 : for any extreme cell, diagonal sum, in the plane partition = rectangular sum, in the multiset.
- order constraint, size constraint
- dynamic programming

simple algorithm, but difficult proof!

Pak's bijection – illustrated example



Application of Pak's algorithm on an example.

Pak's algorithm

Input : a diagram D of a multiset in $\mathcal{M}$.
Output: a plane partition.
Let ℓ be the length and w be the width of D.
for $i := \ell - 1$ **downto** 0 **do**
    for $j := w - 1$ **downto** 0 **do**
        $D[i,j] \leftarrow D[i,j] + \max(D[j+1,i]), D[i,j+1]);$
        for $c := 1$ **to** $\min(w-1-j, \ell-1-i)$ **do**
            $x \leftarrow i + c\,;\ y \leftarrow j + c;$
            $D[x,y] \leftarrow \max(D[x+1,y], D[x,y+1])\ ;$
                    $+\min(D[x+1,y], D[x,y+1]);$
                    $-D[x,y];$
Return D;

Boltzmann sampler

Random sampling under Boltzmann model

- for any constructible class
- approximate size sampling,
- size distribution spread over the whole combinatorial class, but uniform for a sub-class of objects of the same size,
- control parameter,
- automatized sampling : the sampler is compiled from specification automatically,
- very large objects can be sampled.

Model definition

Definition

In the unlabelled case, Boltzmann model assigns to any object $c \in \mathcal{C}$ the following probability :

$$\mathbb{P}_x(c) = \frac{x^{|c|}}{C(x)}$$

A Boltzmann sampler $\Gamma C(x)$ for the class $\mathcal{C}$ is a process that produces objects from $\mathcal{C}$ according to this model.

→ 2 object of the same size will be drawn with the same probability.

The probability of drawing an object of size N is then :

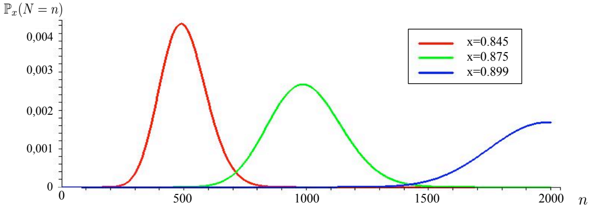
$$\mathbb{P}_x(N = n) = \sum_{|c|=n} \mathbb{P}_x(c) = \frac{C_n x^n}{C(x)}$$

Then, the expected size of an object drawn by a generator with parameter x is :

$$\mathbb{E}_x(N) = x\frac{C'(x)}{C(x)}$$

Approximate and exact-size samplers

- Free samplers : produce objects with randomly varying sizes !
- Tuned samplers : choose x so that expected size is n.
- Run the targeted sampler until the output size is in the desired range (rejection).
- Size distribution of free sampler determines complexity.



Unions, products, sequences

Disjoint unions

Boltzmann sampler ΓC for $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$:

With probability $\frac{A(x)}{C(x)}$ do $\Gamma A(x)$ else do $\Gamma B(x)$ $\longrightarrow$ Bernoulli.

Products

Boltzmann sampler ΓC for $\mathcal{C} = \mathcal{A} \times \mathcal{B}$:

Generate a pair $\langle\ \Gamma A(x)\ ,\ \Gamma B(x)\ \rangle$ $\longrightarrow$ independent calls.

Sequences

Boltzmann sampler ΓC for $\mathcal{C} = \textsc{Seq}(\mathcal{A})$:

Generate k according to a geometric law of parameter $A(x)$

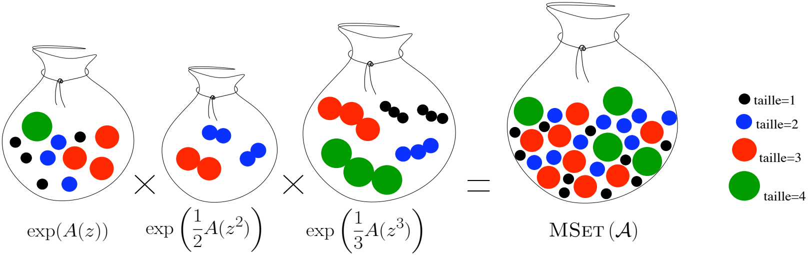
Generate a k-tuple $\langle\ \Gamma A(x)\ , \ldots ,\ \Gamma A(x)\ \rangle$ $\longrightarrow$ independent calls.

Remark : $A(x)$, $B(x)$, and $C(x)$ is given by an ***oracle***.

Generating multisets

$$\mathcal{C} = \text{MSet}(\mathcal{A}) \cong \prod_{\gamma \in \mathcal{A}} \text{Seq}(\gamma) \quad \Rightarrow \quad C(z) = \prod_{\gamma \in \mathcal{A}} (1 - z^{|\gamma|})^{-1}$$

$$C(z) = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} A(z^k) \right) = \prod_{k=1}^{\infty} \exp\left(\frac{1}{k} A(z^k) \right)$$



Sampling an object of $\mathcal{M}$

Algorithm $\Gamma M(x)$

M is the diagram of the multiset to be generated

- Draw m, the max. index of a subset, depending on x ;
- For each index k of a subset until $m-1$
  - Draw the number p of elements to sample, according to a Poisson law of parameter $\frac{x^k}{k(1-x^k)^2}$.
  - Perform p calls to the sampler for $\mathcal{Z} \times \textsc{Seq}(\mathcal{Z})^2$ with parameter x^k, and each time, add k copies of the result to the multiset.

    Repeat p times :

    $i \leftarrow \mathrm{Geom}(x^k)$;
    $j \leftarrow \mathrm{Geom}(x^k)$;
    $M[i,j] \leftarrow M[i,j] + k$
- for index m, draw the number p of elements to generate, according to a non zero Poisson law.

Sampling $\mathcal{M}_{a,b}$ an $\mathcal{M}_D$

$\Gamma M_{a,b}(x)$ [Boltzmann sampler for $\mathcal{M}_{a,b}$]

```
M is the diagram of the multiset to be generated
for i ← 0 to a − 1 do
    for j ← 0 to b − 1 do
        M[i, j] ← Geom(x^{i+j+1});
return M;
```

$\Gamma S_D(x)$ [Boltzmann sampler for $\mathcal{M}_D$]

```
M is the diagram of the multiset to be generated
for (i, j) ∈ D do
    M[i, j] ← Geom(x^{i+j+1});
return M;
```

! the free Boltzmann samplers operate in linear time in the size of the bounding rectangle of the diagram produced.

Summary

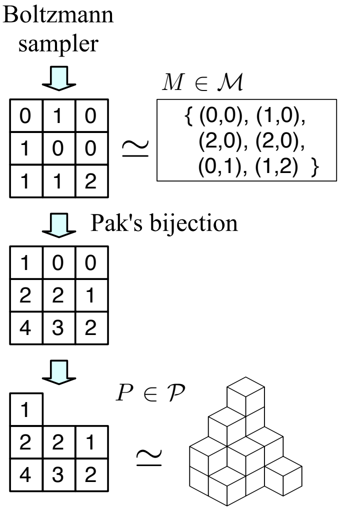
- Targeted Boltzmann sampler for
  - $\mathcal{M}$ → plane partitions
  - $\mathcal{M}_{a,b}$ → boxed plane partitions
  - $\mathcal{S}_D$ → skew planes partitions

  Output : a *diagram* D.

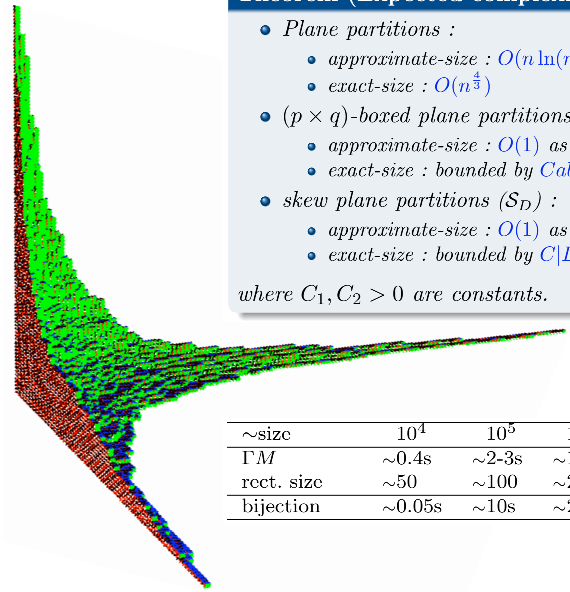
- Rejection

- Pak's algorithm transforms D into a plane partition.

- Size of the output plane partition = size of the original diagram.



Results



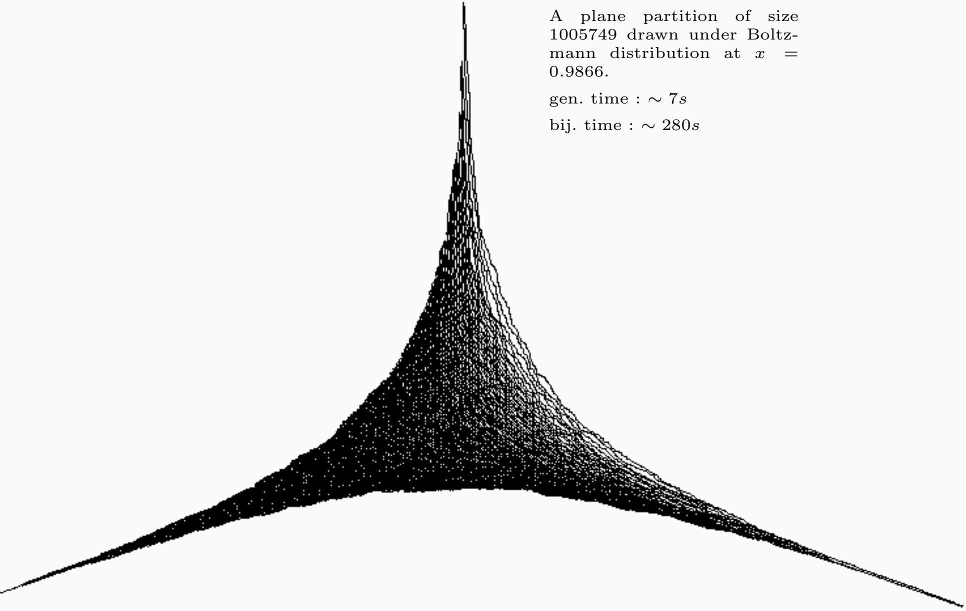
Theorem (Expected complexity)

- *Plane partitions :*
  - *approximate-size :* $O(n \ln(n)^3)$
  - *exact-size :* $O(n^{\frac{4}{3}})$
- $(p \times q)$*-boxed plane partitions (for fixed* a, b*) :*
  - *approximate-size :* $O(1)$ *as* $n \to \infty$
  - *exact-size : bounded by* $Cab.n$
- *skew plane partitions* $(\mathcal{S}_D)$ *:*
  - *approximate-size :* $O(1)$ *as* $n \to \infty$
  - *exact-size : bounded by* $C|D|.n$

where $C_1, C_2 > 0$ *are constants.*

∼size	10^4	10^5	10^6	10^7
ΓM	∼0.4s	∼2-3s	∼10s	∼60s
rect. size	∼50	∼100	∼200-300	∼600-800
bijection	∼0.05s	∼10s	∼20s	∼250-300s

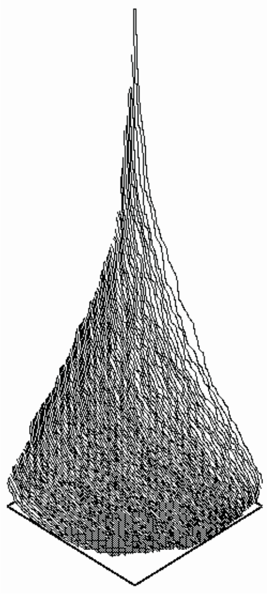
Results – 2



A plane partition of size 1005749 drawn under Boltzmann distribution at $x = 0.9866$.

gen. time : $\sim 7s$

bij. time : $\sim 280s$



← A (100×100)-boxed plane partition of size 999400 drawn under Boltzmann distribution at $x = 0.9931$.

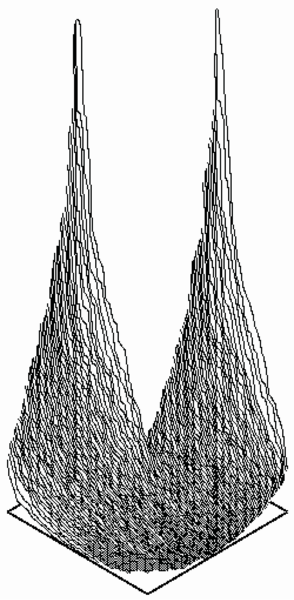
gen. time : $\sim 5s$

bij. time : $\sim 0.7s$

→ A skew plane partition of size 1005532 on the index-domain :
$[0..99] \times [0..99] \setminus [0..49] \times [0..49]$, drawn under Boltzmann distribution at $x = 0.9942$.

gen. time : $\sim 4s$.

bij. time : $\sim 0.35s$.



Analysis of Complexity

Theorem (Expected complexity)

- *Plane partitions :*
  - *approximate-size :* $O(n \ln(n)^3)$
  - *exact-size :* $O(n^{\frac{4}{3}})$
- $(p \times q)$*-boxed plane partitions (for fixed* a, b*) :*
  - *approximate-size :* $O(1)$ *as* $n \to \infty$
  - *exact-size : bounded by* $Cab.n$
- *skew plane partitions (*$\mathcal{S}_D$*) :*
  - *approximate-size :* $O(1)$ *as* $n \to \infty$
  - *exact-size : bounded by* $C|D|.n$

where $C_1, C_2 > 0$ *are constants.*

General scheme

Generation of a plane partition of size n (resp. $\sim n$), with a targeted sampler, i.e., with a parameter tuned such that $\mathbb{E}(N_x) = n$.

mean cost

$$=$$

cost of one call to ΓM $\times$ expected number of calls

$$+$$

cost of Pak's algorithm

1. cost of one call to ΓM : $O(n^{\frac{2}{3}})$
2. expected number of calls to the sampler :
   - approximate size sampler : $O(1)$
   - exact size sampler : $O(n^{\frac{2}{3}})$
3. expected complexity of Pak's algorithm applied to a diagram of size n : $O(n \ln(n)^3)$

Details – free sampler

complexity of the free Boltzmann sampler, as $x \to 1^-$:

$$\Lambda P(x) = \Lambda M(x) + \mathbb{E}_x[\text{PakAlgo}](x)$$

$$\Lambda M(x) = \sum_{i \geq 1} \mathbb{E}\left(\text{Pois}\left(\frac{A(x^i)}{i}\right)\right)\Lambda A(x^i) = \sum_{i \geq 1} \frac{A(x^i)}{i}\Lambda A(x^i)$$

using Mellin transform :

$$\Lambda M(x) = \underset{x \to 1^-}{\mathcal{O}}\left(\frac{1}{(1-x)^2}\right)$$

length of the bounding rectangle of a multiset drawn under Boltzmann model : $\mathcal{O}((1-x)^{-1}\ln((1-x)^{-1}))$ as $x \to 1^-$:

$$\mathbb{E}_x[\text{PakAlgo}](x) = \underset{x \to 1^-}{\mathcal{O}}\left(\frac{1}{(1-x)^3}\ln\left(\frac{1}{1-x}\right)^3\right) = \Lambda P(x)$$

Details – targeted sampler

using Mellin transform :

$$\mathbb{E}(N_x) = \frac{2\zeta(3)}{(1-x)^3} + \underset{x\to 1^-}{\mathcal{O}}\left(\frac{1}{(1-x)^2}\right)$$
$$\mathbb{V}(N_x) = \frac{6\zeta(3)}{(1-x)^4} + \underset{x\to 1^-}{\mathcal{O}}\left(\frac{1}{(1-x)^3}\right)$$

tuned parameter : $\xi_n := 1 - (2\zeta(3)/n)^{1/3}$

expected complexity of $\Gamma M(\xi_n)$ and Pak's algorithm under the uniform distribution at a *fixed* size n :

$$\Lambda M(\xi_n) = \mathcal{O}(n^{\frac{2}{3}}), \qquad \mathbb{E}_n[\text{Pak}] = \mathcal{O}(n \log(n)^3)$$

probability that the output of $\Gamma P(\xi_n)$ has size n :

- using Chebyshev inequality : $\pi_{n,\epsilon} \underset{n\to\infty}{\to} 1$
- using Mellin transform and the saddle-point method :

$$\pi_n \underset{n\to\infty}{\sim} \frac{c}{n^{2/3}}, \text{ with } c \approx 0.1023$$

Details – boxed, skew

sampler for $(a \times b)$-boxed plane partitions :

$$\xi_n^{a,b} := 1 - ab/n$$

$$\pi_{n,\epsilon} \underset{n\to\infty}{\sim} \mathcal{O}(1), \qquad \pi_n \sim \mathcal{O}(n)$$

$\Gamma P_{a,b}(x)$ is of constant complexity $C \cdot a \cdot b$

expected complexity of the approximate-size sampler :

$$\Lambda P_{a,b}(\xi_n)/\pi_{n,\epsilon} \sim C \cdot ab$$

expected complexity of the exact-size sampler :

$$\Lambda P_{a,b}(\xi_n)/\pi_n \sim Cabn$$

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