Plane Partitions: MacMahon's Dream Has Come True

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Outline of the Talk

- Introduction to MacMahon's Partition Analysis and to Omega
- MacMahon's Dream
- Why Did MacMahon Fail? – A Computational Speculation
- Connections to Stanley and Gansner
- The Crucial Rational Functions $Q_{\mathbb{A}}^{\mathbb{X}}$
- The Results Summarized
- The Fundamental Step in the Proof
- Conclusion

Reference: G.E. Andrews and PP, "MacMahon's Partition Analysis XII: Plane Partitions", J. London Math. Soc., 2007.

MacMahon's Partition Analysis and the Omega P

Loading the Omega Package

```
SetDirectory["/home/ppaule/RISC_Comb_Software_Sep05.dir/Omega/"]
```

```
/home/ppaule/RISC_Comb_Software_Sep05.dir/Omega
```

```
<< Omega2.m
```

```
Omega Package by Axel Riese (in cooperation with George E. Andrews and Peter Paule) — © RIS
2.47 (06/21/05)
```

Software of RISC Combinatorics Group

Freely available at: http://www.risc.uni–linz.ac.at/research/combinat/software

Triangles with sides of integer length

PROBLEM (e.g., R.Stanley,1986): Let $t(n)$ be the number of non-congruent triangles with sides of integer length and with perimeter n. Find

$$T(q) := \sum_{n=3}^{\infty} t(n)\, q^n$$

Example: $t(9) = 3$ corresponding to $1+4+4$, $2+3+4$, and $3+3+3$.

$$T(q) = \sum_{\substack{a,b,c\geq 1 \\ a\leq b\leq c,\ a+b>c}} q^{a+b+c} = ?$$

$$T(q) = \sum_{\substack{a,b,c\geq 1\\ a\leq b\leq c,\ a+b>c}} q^{a+b+c}$$

$$= \underset{\geq}{\Omega} \sum_{a,b,c\geq 1} \lambda_1^{b-a} \lambda_2^{c-b} \lambda_3^{a+b-c-1} q^{a+b+c}$$

$$= \underset{\geq}{\Omega} \frac{q^3}{\left(1 - \frac{q\lambda_2}{\lambda_3}\right)\left(1 - \frac{q\lambda_3}{\lambda_1}\right)\left(1 - \frac{q\lambda_1\lambda_3}{\lambda_2}\right)}$$

In such situations MacMahon eliminated the λ by applying successively basic elimination rules such as

$$\underset{\geq}{\Omega} \frac{1}{(1 - x\lambda)\left(1 - \frac{y}{\lambda}\right)} = \frac{1}{(1-x)(1-xy)}.$$

REMARK: MacMahon's Omega Operator $\underset{\geq}{\Omega}$:

$$\underset{\geq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\ldots,s_r}.$$

Recall:

$$\underset{\geq}{\Omega} \frac{1}{(1 - x\,\lambda)\left(1 - \frac{y}{\lambda}\right)} = \frac{1}{(1 - x)(1 - x\,y)}.$$

This way one finds that

$$\begin{aligned}
T(q) &= \underset{\geq}{\Omega} \frac{q^3}{\left(1 - \frac{q\,\lambda_2}{\lambda_3}\right)\left(1 - \frac{q\,\lambda_3}{\lambda_1}\right)\left(1 - \frac{q\,\lambda_1\,\lambda_3}{\lambda_2}\right)} \\
&= \underset{\geq}{\Omega} \frac{q^3}{\left(1 - \frac{q}{\lambda_3}\right)\left(1 - \frac{q\,\lambda_3}{\lambda_1}\right)(1 - q^2\,\lambda_1)} \\
&= \underset{\geq}{\Omega} \frac{q^3}{\left(1 - \frac{q}{\lambda_3}\right)(1 - q^3\,\lambda_3)(1 - q^2)} \\
&= \frac{q^3}{(1 - q^4)(1 - q^3)(1 - q^2)}
\end{aligned}$$

With the package Omega all steps are carried out automatically:

```
OSum[q^(a+b+c), {1 ≤ a, 1 ≤ b, 1 ≤ c, a ≤ b, b ≤ c, a + b > c}, λ]
```

$$\underset{\lambda_1,\lambda_2,\lambda_3}{\underset{\geq}{\Omega}} \frac{q^3}{\left(1-\frac{q\,\lambda_2}{\lambda_3}\right)\left(1-\frac{q\,\lambda_3}{\lambda_1}\right)\left(1-\frac{q\,\lambda_1\,\lambda_3}{\lambda_2}\right)}$$

```
T3 = OR[%]
```

Eliminating λ_2...

Eliminating λ_3...

Eliminating λ_1...

$$\frac{q^3}{(1-q^2)(1-q^3)(1-q^4)}$$

The full generating function generates all the triangles:

```
OSum[x^a y^b z^c q^(a+b+c), {1 ≤ a, 1 ≤ b, 1 ≤ c, a ≤ b, b ≤ c, a + b > c}, λ]
```

$$\underset{\lambda_1,\lambda_2,\lambda_3}{\underset{\geq}{\Omega}} \frac{q^3 x y z}{\left(1 - \frac{q z \lambda_2}{\lambda_3}\right)\left(1 - \frac{q x \lambda_3}{\lambda_1}\right)\left(1 - \frac{q y \lambda_1 \lambda_3}{\lambda_2}\right)}$$

```
OR[%]
```

Eliminating λ_2...

Eliminating λ_3...

Eliminating λ_1...

$$\frac{q^3 x y z}{(1 - q^2 y z)(1 - q^3 x y z)(1 - q^4 x y z^2)}$$

```
Series[%, {x, 0, 4}, {
   y, 0, 4}, {z, 0, 4}, {q, 0, 10}];
```

```
Expand[Coefficient[%, q^9]]
```

$$x^3 y^3 z^3 + x^2 y^3 z^4 + x y^4 z^4$$

n–Gons with sides of integer length (US navy problem)

Let us try to generalize the triangle problem:

```
OSum[q^(a+b+c+d), {1 ≤ a, 1 ≤ b, 1 ≤ c, 1 ≤ d,
        a ≤ b, b ≤ c, c ≤ d, a + b + c > d}, λ]
```

$$\underset{\lambda_1,\lambda_2,\lambda_3,\lambda_4}{\overset{\Omega}{\geq}} \frac{q^4 \lambda_4}{\left(1-\frac{q\lambda_3}{\lambda_4}\right)\left(1-\frac{q\lambda_4}{\lambda_1}\right)\left(1-\frac{q\lambda_1\lambda_4}{\lambda_2}\right)\left(1-\frac{q\lambda_2\lambda_4}{\lambda_3}\right)}$$

```
T4 = OR[%]
```

Eliminating λ_3...

Eliminating λ_2...

Eliminating λ_1...

Eliminating λ_4...

$$-\frac{q^4\left(-1+q^2-q^3\right)}{(1-q)\left(1-q^2\right)\left(1-q^4\right)\left(1-q^6\right)}$$

```
OSum[q^(a+b+c+d+e), {1 ≤ a, 1 ≤ b, 1 ≤ c, 1 ≤ d, 1 ≤ e,
        a ≤ b, b ≤ c, c ≤ d, d ≤ e, a + b + c + d > e}, λ]
```

```
T5 = OR[%]
```

```
OSum[q^(a+b+c+d+e+f),
 {1 ≤ a, 1 ≤ b, 1 ≤ c, 1 ≤ d, 1 ≤ e, 1 ≤ f,
                  a ≤ b, b ≤ c, c ≤ d, d ≤ e, e ≤ f,
                  a + b + c + d + e > f}, λ]
```

$$\mathrm{OSum}[q^{a+b+c+d+e+f},\ \{1 \le a,\ 1 \le b,\ 1 \le c,\ 1 \le d,\ 1 \le e,\ 1 \le f,\ a \le b,\ b \le c,\ c \le d,\ d \le e,\ e \le f,\ a + b + c + d + e > f\},\ \lambda]$$

T_6 = OR[%]

$\{T_3, T_4, T_5, T_6\}$

From inspection of T_3 , T_4 , T_5 , T_6 it is hard to find any common pattern.

BUT : using subprocedures of Omega and the full generating function,
one can FIND and PROVE the following representation for all $k \geq 3$:

$$T_k = \frac{q^k}{(1-q)(1-q^2)\ldots(1-q^k)} - \frac{q^{2k-2}}{1-q}\,\frac{1}{(1-q^2)(1-q^4)\ldots(1-q^{2k-2})}$$

NOTE : The full generating function has the form ($X_i = x_i \cdots x_k$) :

$$\sum x_1^{a_1} \cdots x_k^{a_k} = \frac{X_1}{(1-X_1)\ldots(1-X_k)} - \frac{X_1 X_k^{k-2}}{1-X_k} \times \frac{1}{(1-X_{k-1})(1-X_{k-2} X_k)(1-X_{k-3} X_k^2)\ldots(1-X_1 X_k^{k-2})}$$

where the summation is over all integers a_i such that $a_k \geq a_{k-1} \geq \ldots \geq a_1 \geq 1$ and $a_1 + \cdots + a_{k-1} > a_k$.

[G.E. Andrews, PP, and A. Riese, "MacMahon's Partition Analysis IX: k–Gon Partitions", Bull. Austral. Math. Soc. 64 (2001), 321–329.]

■ REMARKS

● Omega (resp. Partition Analysis) has been used extensively for mathematical discovery; e.g., compositions, partition diamonds, magic squares, etc.

See the references in:
G.E. Andrews and PP, "MacMahon's Partition Analysis XI:
Broken Diamonds and Modular Forms", Acta Arithm. 126 (2007), 281–294]

● Extensions, related combinatorial studies, and software (Maple):
S. Corteel, A. Fu, G. Han, Q.–H. Hou, A. Lascoux, C. Savage, G. Xin, and others.

● Related developments:
J. Stembridge's posets package; based on R. Stanley's work
("Ordered Structures and Partitions", Memoirs AMS 119, 1972);
LattE (J.A. DeLoera, R. Hemmecke, R. Tanzer, R. Yoshida),
an implementation of work of A. Barvinok and J. Pommersheim
("An algorithmic theory of lattice points in polyhedra", MSRI Publ. 38, 1999).

MacMahon's Dream

The plane partition story begins with the paper:

P.A. MacMahon, "Memoir on the Theory of Partitions of Numbers – Part I",Phil.Trans.187 (1897), 619 – 673.

NOTE 1. Parts II–VII followed.

NOTE 2. The six plane partitions of ***3*** are:

$$3 = 2+1 = 1+1+1 = \begin{array}{c} 2 \\ + \\ 1 \end{array} = \begin{array}{l} 1+1 \\ + \\ 1 \end{array} = \begin{array}{c} 1 \\ + \\ 1 \\ + \\ 1 \end{array}$$

■ Conjecture (pp. 657–658)

‘ The enumeration of the three – dimensional graphs that can be formed with a given number of nodes, corresponding to the regularized partitions of multi – partite numbers of given content, is a weighty problem. I have verified to a high order that the generating function of the complete system is

$$(1-q)^{-1}(1-q^2)^{-2}(1-q^3)^{-3}(1-q^4)^{-4}\ldots \textit{ad inf.},$$

and, so far as my investigations have proceeded, everything tends to confirm the truth of this conjecture.’

■ The first coefficients

```
Series[(1 - q)^-1 (1 - q^2)^-2 (1 - q^3)^-3 (1 - q^4)^-4, {q, 0, 4}]
```

J.W.L. GLAISHER's COMMENT

NOTE. Referee report for the Philosophical Transactions of the Royal Society, June 8, 1896. (Printed with permission of the Royal Society.)

"I don't fancy the paper very much, but it must be printed. I don't care much for a paper on very technical mathematics being published in the Phil. Trans. unless there is something very striking in it. However, it is one of a series, and they are in deep water now and cannot go on much farther. I have made my report because there is no more to be said that it should be published (though the interesting results are the conjectural ones!), the balance being on that side."

MacMahon's Dream

Develop the method of PARTITION ANALYSIS to prove the conjecture.

BUT:
His efforts did not turn out as he had hoped,
and he had to spend nearly 20 years finding an
alternative treatment.

Further details on historical aspects:

G.E. Andrews and PP, "MacMahon's Dream'', Report 2006–26,
SFB F013, 2006. (14 pages, to appear).

Some Illustrating Computations:

Plane Partition Generating Function: 1 row & 4 columns

In[] := $\text{OSum}[x_{11}^{a_{11}}\, x_{12}^{a_{12}}\, x_{13}^{a_{13}}\, x_{14}^{a_{14}},\ \{a_{11} \geq a_{12},\ a_{12} \geq a_{13},\ a_{13} \geq a_{14}\},\ \lambda]$

Out[] = $$\underset{\lambda_1,\lambda_2,\lambda_3}{\underset{\geq}{\Omega}} \frac{1}{(1 - x_{11}\,\lambda_1)\left(1 - \frac{x_{12}\,\lambda_2}{\lambda_1}\right)\left(1 - \frac{x_{14}}{\lambda_3}\right)\left(1 - \frac{x_{13}\,\lambda_3}{\lambda_2}\right)}$$

In[]:=OR[%]

Out[] =

$$\frac{1}{(1 - x_{11})\,(1 - x_{11}\,x_{12})\,(1 - x_{11}\,x_{12}\,x_{13})\,(1 - x_{11}\,x_{12}\,x_{13}\,x_{14})}$$

NOTE. The proof for ***1*** row and ***c*** columns (resp., ***r*** rows and ***1*** column) is trivial.

■ Plane Partition Generating Function: 2 rows & 2 columns

In[] := OSum[$x_{11}{}^{a_{11}}\ x_{12}{}^{a_{12}}\ x_{21}{}^{a_{21}}\ x_{22}{}^{a_{22}}$,
{$a_{11} \geq a_{12}$,
$a_{21} \geq a_{22}$,
$a_{11} \geq a_{21}$,
$a_{12} \geq a_{22}$}, {λ, μ}]

$$\underset{\lambda_1,\lambda_2,\lambda_3,\lambda_4}{\overset{\Omega}{\geq}} \frac{1}{(1 - x_{11}\,\lambda_{1,1}\,\mu_{1,1})\left(1 - \frac{x_{12}\,\mu_{1,2}}{\lambda_{1,1}}\right)} \times \frac{1}{\left(1 - \frac{x_{21}\,\lambda_{2,1}}{\mu_{1,1}}\right)\left(1 - \frac{x_{22}}{\lambda_{2,1}\,\mu_{1,2}}\right)}$$

In[]:=OR[%]

Out[] =

$(1 - x_{11}^2\,x_{12}\,x_{21}) / ((1 - x_{11})\,(1 - x_{11}\,x_{12})\,(1 - x_{11}\,x_{21})\,(1 - x_{11}\,x_{12}\,x_{21})\,(1 - x_{11}\,x_{12}\,x$

NOTE. The form of the full generating functions for 2×2 plane partitions,

Out[] =

$(1 - x_{11}^2 x_{12} x_{21}) / ((1 - x_{11}) (1 - x_{11} x_{12}) (1 - x_{11} x_{21}) (1 - x_{11} x_{12} x_{21}) (1 - x_{11} x_{12}$

is explained as follows. The 2×2 plane partitions are generated over ℕ by:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

BUT:

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

■ Plane Partition Generating Function: 3 rows & 3 columns

```
In[] := OSum[q^(a11+a12+a13+a21+a22+a23+a31+a32+a33),
  {a11 ≥ a12, a12 ≥ a13,
   a21 ≥ a22, a22 ≥ a23,
   a31 ≥ a32, a32 ≥ a33,
   a11 ≥ a21, a21 ≥ a31,
   a12 ≥ a22, a22 ≥ a32,
   a13 ≥ a23, a23 ≥ a33},  λ]
```

Out[] =

$$\underset{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12}}{\underset{\geq}{\Omega}} 1 \Big/ \left((1-q\,\lambda_1\,\lambda_7)\left(1-\frac{q\,\lambda_5}{\lambda_8}\right)\left(1-\frac{q\,\lambda_3\,\lambda_8}{\lambda_7}\right)\left(1-\frac{q\,\lambda_2\,\lambda_9}{\lambda_1}\right)\left(1-\frac{q\,\lambda_6}{\lambda_5\,\lambda_{10}}\right)\left(1-\frac{q\,\lambda_4\,\lambda_{10}}{\lambda_3\,\lambda_9}\right)\left(1-\frac{q\,\lambda_{11}}{\lambda_2}\right)\left(1-\frac{q}{\lambda_6\,\lambda_{12}}\right)\left(1-\frac{q\,\lambda_{12}}{\lambda_4\,\lambda_{11}}\right)\right)$$

Eliminating the λ variables:

```
In[]:=OR[%]
```

Out[] = $$\frac{1}{(1-q)\,(1-q^2)^2\,(1-q^3)^3\,(1-q^4)^2\,(1-q^5)}$$

■ Plane Partition Generating Function: 3 rows & 4 columns

```
In[]:=OSum[...]
```

$$\text{Out}[] = \frac{1}{(1-q)\,(1-q^2)^2\,(1-q^3)^3\,(1-q^4)^3\,(1-q^5)^2\,(1-q^6)}$$

NOTE. Pattern:

$$\begin{pmatrix} q & q^2 & q^3 & q^4 \\ q^2 & q^3 & q^4 & q^5 \\ q^3 & q^4 & q^5 & q^6 \end{pmatrix}$$

Recall, if ***r=3*** and ***c=3***:

$$\text{Out}[] = \frac{1}{(1-q)\,(1-q^2)^2\,(1-q^3)^3\,(1-q^4)^2\,(1-q^5)}$$

MacMahon conjectured for r rows and c columns

Let

$P_{r,c}(n)$:= no. of plane partitions with at most r rows and at most c columns,

then:

$$\sum_{n=0}^{\infty} P_{r,c}(n)\, q^n = \prod_{i=1}^{r} \prod_{j=1}^{c} \left(1 - q^{i+j-1}\right)^{-1} .$$

NOTE. On page 187 of "Combinatory Analysis, Vol.II", MacMahon says:

"Our knowledge of the Ω operation is not sufficient to establish the final form of result. This will by the aid of new ideas which will be brought forward in the following chapters."

→ Why did MacMahon fail when using Partition Analysis?

Why did MacMahon fail?
A computational speculation

- **Full Generating Function: 3 rows & 3 columns**

```
Crude33 =
 OSum[x11^a11 x12^a12 x13^a13
      x21^a21 x22^a22 x23^a23
      x31^a31 x32^a32 x33^a33,
   {a11 ≥ a12, a12 ≥ a13,
    a21 ≥ a22, a22 ≥ a23,
    a31 ≥ a32, a32 ≥ a33,
    a11 ≥ a21, a21 ≥ a31,
    a12 ≥ a22, a22 ≥ a32,
    a13 ≥ a23, a23 ≥ a33},
    λ]

FullGenFunction3by3 = Factor[OR[Crude33]]
```

BUT:

let us apply the following substitution of variables:

```
Subst = {x11^p_. → x0^p, x12^p_. → x1^p, x13^p_. → x2^p,    x21^p_. → x-1^p, x22^p_. → x0
  x23^p_. → x1^p,
  x31^p_. → x-2^p, x32^p_. → x-1^p, x33^p_. → x0^p };

Factor[FullGenFunction3by3 /. Subst]
```

Connections to Stanley and Gansner

This way we were led to a rediscovery of a theorem by Emden R. Gansner (1981).

[E.R. Gansner "The enumeration of plane partititions via the Burge correspondence", Illinois J.Math. **25** (1981), 533–554]

NOTE. The pattern of the substitution is $x_{i,j} \to x_{j-i}$; e.g., for $r=c=4$:

$$(x_{i,j}) \to \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_{-1} & x_0 & x_1 & x_2 \\ x_{-2} & x_{-1} & x_0 & x_1 \\ x_{-3} & x_{-2} & x_{-1} & x_0 \end{pmatrix}.$$

R. Stanley's TRACE THEOREM (1973) corresponds to the special case:

$$(x_{i,j}) \to \begin{pmatrix} x & q & q & q \\ q & x & q & q \\ q & q & x & q \\ q & q & q & x \end{pmatrix}.$$

Stanley's Trace Theorem:

Let

$T_{r,c}(t;n)$:= no. of plane partitions of n with at most r rows and at most c columns, and with trace t,

then:

$$\sum_{n=0}^{\infty} T_{r,c}(t;n)\, q^n = \prod_{i=1}^{r} \prod_{j=1}^{c} \left(1 - x\, q^{i+j-1}\right)^{-1} .$$

EXAMPLE: $t=8=t_0$ (and $t_1 = 5$, $t_2 = 3$, $t_{-1} = 6$, etc.) for

4	3	2	2
4	3	1	1
2	2	1	1
1	1	.	.

Gansner's Generalization:

Let

$T_{r,c}(t_{-r+1}, \ldots, t_{-1}; t_0, \ldots, t_{c-1}; n)$
:= no. of plane partitions of n with at most r rows
and at most c columns, and with i - trace t_i,

then:

$$\sum_{n \geq 0} \sum_{t_{-r+1} \geq 0} \cdots \sum_{t_{c-1} \geq 0} q^n T_{r,c}(t_{-r+1}, \ldots, t_{-1}; t_0, \ldots, t_{c-1}; n) \times x_{-r+1}^{t_{-r+1}} \cdots x_{-1}^{t_{-1}} x_0^{t_0} \cdots x_{c-1}^{t_{c-1}} = \prod_{i=1}^{r} \prod_{j=1}^{c} (1 - x_{-r+1} \cdots x_{-1} x_0 \cdots x_{c-1} q^{i+j-1})^{-1} .$$

Generalization of Gansner's result:

$T_{r,c}(t_{-r+1}, \ldots, t_{-1}; t_0, \ldots, t_{c-1}; a_0, \ldots, a_{r-1}; n) :=$
no. of plane partitions of n with at most r rows and at most c columns,
with i - trace t_i, and with $a_{1,c} = a_0, \ldots, a_{r,c} = a_{r-1}$:

$$\sum_{n\geq 0} \sum_{t_{-r+1}\geq 0} \cdots \sum_{t_{c-1}\geq 0} \sum_{a_0\geq a_1\geq a_2\geq\ldots\geq a_{r-1}\geq 0} q^n \times T_{r,c}(t_{-r+1}, \ldots, t_{-1}; t_0, \ldots, t_{c-1}; a_0, \ldots, a_{r-1}; n) \times x_{-r+1}^{t_{-r+1}} \cdots x_{-1}^{t_{-1}} x_0^{t_0} \cdots x_{c-1}^{t_{c-1}} z_0^{a_0} \cdots z_{r-1}^{a_{r-1}}$$

$$= \prod_{i=1}^{r} \prod_{j=1}^{c-1} (1 - x_{-i+1} \cdots x_{-1} x_0 \cdots x_{j-1} q^{i+j-1})^{-1} \times Q_{\mathbb{B}}^{\mathbb{Y}}\left(q\, x_{c-1} z_0, \frac{z_1}{Y_0}, \frac{z_2}{Y_1}, \ldots, \frac{z_{r-1}}{Y_{r-2}}\right).$$

where $\mathbb{Y} = \{Y_0, Y_1, \ldots, Y_{r-2}\}$, $\mathbb{B} = \{Y_{c-1}, Y_c, \ldots, Y_{c+r-2}\}$, and $Y_k = q^k x_{c-k-1} x_{c-k} \cdots x_{c-2}$.

The Crucial Rational Functions $Q_{\mathbb{A}}^{\mathbb{X}}$

- **Lagrange Symmetrization** (e.g., A. Lascoux [AMS CBMS-99, 2003])

Given: $f = f_{\{A_0, A_1\}}(z) \in K(A_0, A_1, z)$ symmetric in A_0 and A_1.

From f one obtains a rational function $L_{\mathbb{A}}(f)$ that is symmetric in all the variables from $\mathbb{A}=\{A_0, A_1, A_2\}$ by

$$L_{\mathbb{A}}(f) := \sum_{i=0}^{2} \frac{f_{\mathbb{A}\setminus\{A_i\}}(A_i)}{\prod_{A'\in\mathbb{A}\setminus\{A_i\}}(A_i - A')} \in K(A_0, A_1, A_2).$$

This means,

$$L_{\mathbb{A}}(f) =$$

$$\frac{f_{\{A_1,A_2\}}(A_0)}{(A_0 - A_1)(A_0 - A_2)} + \frac{f_{\{A_0,A_2\}}(A_1)}{(A_1 - A_0)(A_1 - A_2)} + \frac{f_{\{A_0,A_1\}}(A_2)}{(A_2 - A_0)(A_2 - A_1)}$$

■ **Definition of $Q_{\mathbb{A}}^{\mathbb{X}}$** $(\in \mathbb{Q}(\mathbb{A}, \mathbb{X}, z_0, \ldots, z_n))$

For $n \geq 0$ let $\mathbb{A} = \{A_0, \ldots, A_n\}$, $\mathbb{X} = \{X_0, \ldots, X_{n-1}\}$:

$$n = 0: \qquad Q_{\{A_0\}}^{\emptyset}(z_0) := \frac{1}{1 - A_0 z_0};$$

$$n \geq 1: \qquad Q_{\mathbb{A}}^{\mathbb{X}}(z_0, \ldots, z_{n-1}, z_n) := \frac{(-1)^n A_0 \cdots A_{n-1} A_n}{1 - A_0 \cdots A_{n-1} A_n z_0 \cdots z_{n-1} z_n} L_{\mathbb{A}}(f)$$

where for $i \in \{0,\ldots,n\}$:

$$f_{\mathbb{A} \setminus \{A_i\}}(z) := \frac{1}{z} \prod_{j=0}^{n-1} \left(1 - \frac{z}{X_j}\right) Q_{\mathbb{A} \setminus \{A_i\}}^{\mathbb{X} \setminus \{X_{n-1}\}}(z_0, \ldots, z_{n-1}).$$

NOTE. Similarly we define rational functions

$$R_{\mathbb{A}}^{\mathbb{X}}(w_0, \ldots, w_n; z_0, \ldots, z_n).$$

NOTE. Both $Q_{\mathbb{A}}^{\mathbb{X}}$ and $R_{\mathbb{A}}^{\mathbb{X}}$ are symmetric in all the $\mathbb{A}$ variables.

■ A crucial evaluation:

$$Q_{\mathbb{A}}^{\mathbb{X}}(0, z_1, \ldots, z_n) = 1.$$

Proof. For the induction step $n-1 \to n$ we need to show:

$$\sum_{i=0}^{n} \frac{f_{\mathbb{A}\setminus\{A_i\}}(A_i)}{\prod_{A'\in\mathbb{A}\setminus\{A_i\}}(A_i - A')} = \frac{(-1)^n}{A_0 \cdots A_n}$$

where

$$f_{\mathbb{A}\setminus\{A_i\}}(A_i) = \frac{1}{z} \prod_{j=0}^{n-1} \left(1 - \frac{A_i}{X_j}\right).$$

After expanding this product, the assertion is implied by Lagrange interpolation:

$$\sum_{i=0}^{n} \frac{A_i^k}{\prod_{A'\in\mathbb{A}\setminus\{A_i\}}(A_i - A')} = \begin{cases} 0, & \text{if } 0 \leqslant k \leqslant n-1 \\ \frac{(-1)^n}{A_0 \cdots A_n}, & \text{if } k = -1 \end{cases}.$$

■ A crucial $\underset{\geq}{\Omega}$ elimination

For $n \geq 0$ let $\mathbb{A} = \{A_0, \ldots, A_n\}$, $\mathbb{X} = \{X_0, \ldots, X_{n-1}\}$;
A_{n+1}, $z_1, \ldots, z_n$ and $w_0, \ldots, w_n$, w_{n+1} are additional variables :

$$\underset{\geq}{\Omega} \frac{Q_{\mathbb{A}}^{\mathbb{X}}(w_0 \lambda_0, \ldots, w_n \lambda_n)}{1 - A_0 \cdots A_n A_{n+1} w_0 \cdots w_n w_{n+1} \lambda_0 \cdots \lambda_n} \prod_{k=0}^{n} \left(1 - \frac{z_0 \cdots z_k}{\lambda_0 \cdots \lambda_k}\right)^{-1}$$
$$= \frac{1}{1 - A_{n+1} w_{n+1}} \left(R_{\mathbb{A}}^{\mathbb{X}}(w_0, \ldots, w_n; z_0, \ldots, z_n) - A_{n+1} w_{n+1} R_{\mathbb{A}}^{\mathbb{X}}(w_0, \ldots, w_{n-1}, w_n w_{n+1} A_{n+1}; z_0, \ldots, z_n)\right)$$

Proof. By elementary $\underset{\geq}{\Omega}$ elimination rules.

The Results Summarized

Notation : $$p_{m,n}\begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ x_{2,1} & \cdots & x_{2,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{pmatrix}$$

$$:= \sum_{(a_{i,j}) \in \mathcal{P}_{m,n}} x_{1,1}^{a_{1,1}} \cdots x_{1,n}^{a_{1,n}} \, x_{2,1}^{a_{2,1}} \cdots x_{2,n}^{a_{2,n}} \times \cdots \times x_{m,1}^{a_{m,1}} \cdots x_{m,n}^{a_{m,n}}$$

where

$$(a_{i,j}) = \begin{pmatrix} a_{1,1} & \to & a_{1,2} & \to & \cdots & \to & a_{1,n} \\ \downarrow & & \downarrow & & & & \downarrow \\ a_{2,1} & \to & a_{2,2} & \to & \cdots & \to & a_{2,n} \\ \downarrow & & \downarrow & & & & \downarrow \\ \vdots & & \vdots & & \ddots & & \vdots \\ \downarrow & & \downarrow & & & & \downarrow \\ a_{m,1} & \to & a_{m,2} & \to & \cdots & \to & a_{m,n} \end{pmatrix} \in \mathcal{P}_{m,n}$$

Notation: $X_0 := 1, \quad X_k := x_1 \cdots x_k \quad (k \geqslant 1)$

Theorem: *For* ***m,n≥0***:

$$p_{m+1,n+1}\begin{pmatrix} x_n & x_{n-1} & x_{n-2} & \cdots & x_1 & z_0 \\ x_{n+1} & x_n & x_{n-1} & \cdots & x_2 & z_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n+m} & x_{n+m-1} & x_{n+m-2} & \cdots & x_{m+1} & z_m \end{pmatrix}$$

$$= \prod_{k=0}^{n-1} \left(1 - \frac{X_n}{X_k}\right)^{-1} \left(1 - \frac{X_{n+1}}{X_k}\right)^{-1} \cdots \left(1 - \frac{X_{n+m}}{X_k}\right)^{-1}$$

$$\times\, Q^{\{X_0, \ldots, X_{m-1}\}}_{\{X_n, \ldots, X_{n+m}\}}\left(\frac{z_0}{X_0}, \frac{z_1}{X_1}, \ldots, \frac{z_m}{X_m}\right).$$

NOTE. The matrix in the x_i is of Toeplitz shape!

In the special case: $z_0 := 0$ the *Theorem* turns into:

$$p_{m+1,n+1}\begin{pmatrix} x_n & x_{n-1} & x_{n-2} & \cdots & x_1 & 0 \\ x_{n+1} & x_n & x_{n-1} & \cdots & x_2 & z_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n+m} & x_{n+m-1} & x_{n+m-2} & \cdots & x_{m+1} & z_m \end{pmatrix}$$

$$= \prod_{k=0}^{n-1} \left(1 - \frac{X_n}{X_k}\right)^{-1} \left(1 - \frac{X_{n+1}}{X_k}\right)^{-1} \cdots \left(1 - \frac{X_{n+m}}{X_k}\right)^{-1}$$

$$\times\ Q^{\{X_0, \ldots, X_{m-1}\}}_{\{X_n, \ldots, X_{n+m}\}}\left(\frac{0}{X_0}, \frac{z_1}{X_1}, \ldots, \frac{z_m}{X_m}\right),$$

and we obtain the following corollary:

Recall the notation: $X_0 := 1$, $X_k := x_1 \cdots x_k \quad (k \geqslant 1)$

Corollary: *For* $m \geq 0$ *and* $n \geq 1$*:*

$$p_{m+1,n}\begin{pmatrix} x_n & x_{n-1} & x_{n-2} & \cdots & x_1 \\ x_{n+1} & x_n & x_{n-1} & \cdots & x_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n+m} & x_{n+m-1} & x_{n+m-2} & \cdots & x_{m+1} \end{pmatrix}$$

$$= \prod_{k=0}^{n-1} \left(1 - \frac{X_n}{X_k}\right)^{-1} \left(1 - \frac{X_{n+1}}{X_k}\right)^{-1} \cdots \left(1 - \frac{X_{n+m}}{X_k}\right)^{-1}.$$

NOTE. This is equivalent to E.R. Gansner's theorem (1981). – The generalization mentioned previously in this talk is the following special case of our *Theorem:*

Notation: $Y_0 := 1, \quad Y_k := q^k x_{c-k} \cdots x_{c-1} \quad (k \geqslant 1)$

Corollary: For $r,c \geqslant 0$:

$$p_{r+1,c+1}\begin{pmatrix} q\,x_0 & q\,x_1 & q\,x_2 & \cdots & q\,x_{c-1} & q\,x_c\,z_0 \\ q\,x_{-1} & q\,x_0 & q\,x_1 & \cdots & q\,x_{c-2} & q\,x_{c-1}\,z_1 \\ q\,x_{-2} & q\,x_{-1} & q\,x_0 & \cdots & q\,x_{c-3} & q\,x_{c-2}\,z_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q\,x_{-r} & q\,x_{-r+1} & q\,x_{-r+2} & \cdots & q\,x_{c-r-1} & q\,x_{c-r}\,z_r \end{pmatrix}$$

$$= \prod_{i=1}^{r+1}\prod_{j=0}^{c}\frac{1}{1 - x_{-i+1}\cdots x_{j-1}\,q^{i+j-1}}$$

$$\times\ Q^{\{Y_0,\ldots,Y_{r-1}\}}_{\{Y_c,\ldots,Y_{c+r}\}}\left(q\,x_c\,z_0,\ \frac{z_1}{Y_0},\ \frac{z_2}{Y_1},\ \ldots,\ \frac{z_r}{Y_{r-1}}\right)$$

$$= \sum_{n\geq 0}\ \sum_{t_{-r}\geq 0}\cdots\sum_{t_c\geq 0}\ \sum_{a_0\geq a_1\geq a_2\geq\ldots\geq a_r\geq 0} q^n$$

$$\times\ T_{r+1,c+1}\,(t_{-r},\ \ldots,\ t_{-1};\ t_0,\ \ldots,\ t_c;\ a_0,\ \ldots,\ a_r;\ n)$$

$$\times\ x_{-r}^{t_{-r}}\cdots x_{-1}^{t_{-1}}\,x_0^{t_0}\cdots x_c^{t_c}\,z_0^{a_0}\cdots z_r^{a_r}$$

The Fundamental Step in the Proof

- **The Basic Reduction Lemma:** For $m,n\geq 0$:

$$p_{m+1,n+1}\begin{pmatrix} x_{1,1} & \cdots & x_{1,n} & z_0 \\ \vdots & \ddots & \vdots & \vdots \\ x_{m,1} & \cdots & x_{m,n} & z_{m-1} \\ x_{m+1,1} & \cdots & x_{m+1,n} & z_m \end{pmatrix} = \left(1 - z_0 \cdots z_m \prod_{\substack{1\leq i\leq m+1 \\ 1\leq j\leq n}} x_{i,j}\right)^{-1}$$

$$\times \underset{\geq}{\Omega}\, p_{m+1,n}\begin{pmatrix} x_{1,1} & \cdots & x_{1,n-1} & x_{1,n}\,\lambda_0 \\ \vdots & \ddots & \vdots & \vdots \\ x_{m,1} & \cdots & x_{m,n-1} & x_{m,n}\,\lambda_{m-1} \\ x_{m+1,1} & \cdots & x_{m+1,n-1} & x_{m+1,n} \end{pmatrix} \prod_{i=1}^{m}\left(1 - \frac{z_0 \cdots z_{i-1}}{\lambda_0 \cdots \lambda_{i-1}}\right)^{-1}$$

Proof. Immediate from the crude generating function $p_{m+1,n+1} = \underset{\geq}{\Omega}(\cdots)$.

NOTE. Using this lemma, the induction step $n\to n+1$ in the proof of our *Theorem* can be settled b

elimination" (involving $Q_{\mathbb{A}}^{\mathbb{X}}$) shown before.

Conclusion

- Crucial observation: The "discovery" of the substitution $x_{i,j} \to x_{j-i}$ which led to a factorization of the full generating function.
- Partition Analysis provided the first "direct" generating function approach. (Previous proofs used determinants, lattice paths, and/or combinatorial bijections like RSK, Bender–Knuth, Burge, etc.)
- Another ingredient: Lagrange symmetrization (resp. interpolation).
- Open problems: Apply this approach to other plane partition settings. (Properties of $Q_{\mathbb{A}}^{\mathbb{X}}$ and $R_{\mathbb{A}}^{\mathbb{X}}$ need to be explored further.)