Product of Linear Differential Operators by Evaluation and Interpolation

Nicolas Le Roux
nicolas.le_roux@inria.fr

INRIA Rocquencourt, Algorithms Project

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Joint work with Alin Bostan and Frédéric Chyzak.

Statement of the Problem

Notations

- $\partial = \dfrac{d}{dX}$, $\mathbb{K}$ a field.
- $\mathbb{K}[X]\langle\partial\rangle$ the ring of skew polynomials in (X, ∂).

Canonical form of $L \in \mathbb{K}[X]\langle\partial\rangle$: $L = \sum_{i=0}^{r_L} \sum_{j=0}^{d_L} l_{i,j} X^j \partial^i$.

(r_L, d_L) is the bidegree of L.

Problem

Given $B, A \in \mathbb{K}[X]\langle\partial\rangle$ in canonical form, compute the canonical form of BA.

Review of Known Algorithms

- Naive Algorithms

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    - Using naive expansion and the Leibniz Formula:

$$BA = \sum_{0 \le i,j,k,l \le n} b_{i,j}\, a_{k,l} X^j \left(\partial^i X^l\right) \partial^k;$$

The canonical form of $\partial^i X^l$ is obtained by Leibniz formula:

$$\partial^i X^l = \sum_{m=0}^{\min(i,l)} (l)_m \binom{i}{m} X^{l-m}\, \partial^{i-m}.$$

This leads to $\mathcal{O}(n^5)$ operations in $\mathbb{K}$.

Review of Known Algorithms

- Naive Algorithms
  - Using naive expansion and the Leibniz Formula: $\mathcal{O}(n^5)$;
  - Using an iterative scheme and polynomial FFT:

  Write B as $B = \sum_{i=0}^{n} b_i(X)\partial^i$ and expand BA as

$$BA = \sum_{i=0}^{n} b_i(X) \left(\partial^i A\right) .$$

  $\partial^i A$ is computed iteratively using the formula

$$\partial L = \frac{dL}{dX} + L\partial.$$

  This leads to $\mathcal{O}(n^2 M(n))$ operations in $\mathbb{K}$.
  $M(n)$ = number of operations in $\mathbb{K}$ for computing the product of two polynomials with degree n.

Review of Known Algorithms

- Naive Algorithms
  - Using naive expansion and the Leibniz Formula: $\mathcal{O}(n^5)$;
  - Using an iterative scheme and polynomial FFT: $\tilde{\mathcal{O}}(n^3)$;
  - Using Takayama's Formula and polynomial FFT:

$$BA = \sum_{k=0}^{n} \frac{1}{k!} \underbrace{\left(\frac{d^k B}{d\partial^k} \star \frac{d^k A}{dX^k} \right)}_{M(n^2)\ ops\ in\ \mathbb{K}}$$

$\star$ means commutative products in bidegree (n, n).
This leads to a $\mathcal{O}(nM(n^2))$ operations in $\mathbb{K}$.

Review of Known Algorithms

- Naive Algorithms
  - Using naive expansion and the Leibniz Formula: $\mathcal{O}(n^5)$;
  - Using an iterative scheme and polynomial FFT: $\tilde{\mathcal{O}}(n^3)$;
  - Using Takayama's Formula and polynomial FFT: $\tilde{\mathcal{O}}(n^3)$;
- van der Hoeven's Algorithm [2002]:
  It reduces the product to a constant number of product of $n \times n$ matrices with entries in $\mathbb{K}$.

Review of Known Algorithms

- Naive Algorithms
  - Using naive expansion and the Leibniz Formula: $\mathcal{O}(n^5)$;
  - Using an iterative scheme and polynomial FFT: $\tilde{\mathcal{O}}(n^3)$;
  - Using Takayama's Formula and polynomial FFT: $\tilde{\mathcal{O}}(n^3)$;
- van der Hoeven's Algorithm [2002]: $\mathcal{O}(n^3)$ and even better.

Results

- Product of dense skew polynomials in $\mathbb{K}[X]\langle\partial\rangle$ with bidegree (n, n).
  - Equivalence with product of $n \times n$ matrices with entries in $\mathbb{K}$;
  - An Evaluation-Interpolation algorithm reducing the number of product of $n \times n$ matrices;
  - An optimal algorithm for product when characteristic of $\mathbb{K}$ is small.
- Further complexity estimate when the skew polynomials are sparse.
- Experiments. Early prototype in Magma more efficient than the existing Magma product.

Results

Number of $n \times n$ Matrix Products in van der Hoeven's algorithm and ours

	vdH	Improved-vdH	Ours
(a, b, c)	(6,4,2)	(6,4,2)	(2,3,2)
Product by blocks	96	48	12
Zero blocks + Strassen product	48	12	8

- Evaluation of B and A resulting in two evaluation matrices of sizes $(an) \times (bn)$ and $(bn) \times (cn)$.
- Computation of the product of the two matrices:

$$a\left\{ \overbrace{}^{b} \right. \times \quad b\left\{ \overbrace{}^{c} \right.$$

Results

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- Evaluation of B and A resulting in two evaluation matrices of sizes $(an) \times (bn)$ and $(bn) \times (cn)$.
- Computation of the product of the two matrices: (abc) products of $n \times n$ matrices.
- Interpolation of C from the $(an) \times (cn)$ matrix.

Steps of van der Hoeven's algorithm

B, A in $\mathbb{K}[X]\langle\partial\rangle$.

- convert them as recurrence operators: this is done by rewriting B and A as elements of $\mathbb{K}[X, X^{-1}]\langle\theta\rangle$;
  - $\theta = X\partial$ plays the role of multiplication by k;
  - X plays the role of the shift operator.

Steps of van der Hoeven's algorithm

B, A in $\mathbb{K}[X]\langle\partial\rangle$.

- convert them as recurrence operators: this is done by rewriting B and A as elements of $\mathbb{K}[X, X^{-1}]\langle\theta\rangle$;
- compute the evaluation matrices of B and A;
- compute the product of the two previous matrices;
- recover $C = BA$ from the last matrix: C is written as an element in $\mathbb{K}[X, X^{-1}]\langle\ \theta\rangle$;
- rewrite C as an element of $\mathbb{K}[X]\langle\partial\rangle$.

Action over $\mathbb{K}[X]$

$\mathbb{K}[X]\langle\theta\rangle$ acts naturally (and faithfully) over $\mathbb{K}[X]$:
$L \in \mathbb{K}[X]\langle\theta\rangle$ defines a $\mathbb{K}$-linear map from $\mathbb{K}[X]$ to itself and is uniquely determined by this map.
Therefore L can be identified with an infinite matrix.

van der Hoeven's observation

Assume L with bidegree (n, n). L is uniquely determined as a $\mathbb{K}$-linear map from $\mathbb{K}[X]_{\leq n}$ to $\mathbb{K}[X]_{\leq 2n}$.

Action over $\mathbb{K}[X]$

van der Hoeven's observation

Assume L with bidegree (n, n). L is uniquely determined as a $\mathbb{K}$-linear map from $\mathbb{K}[X]_{\leq n}$ to $\mathbb{K}[X]_{\leq 2n}$.

Example

$$L = (a + b\,X) + (c + d\,X)\,\theta.$$
$$L(1) = a + b\,X, \quad L(X) = (a + c)\,X + (b + d)\,X^2$$

Associated matrix : $M^L = \begin{bmatrix} a + 0c & 0 \\ b + 0d & a + 1c \\ 0 & b + 1d \end{bmatrix}$

Action over $\mathbb{K}[X]$

van der Hoeven's observation

Assume L with bidegree (n, n). L is uniquely determined as a $\mathbb{K}$-linear map from $\mathbb{K}[X]_{\leq n}$ to $\mathbb{K}[X]_{\leq 2n}$.

Example

$$L(1) = a + b\,X, \quad L(X) = (a + c)\,X + (b + d)\,X^2$$
$$L(X^2) = (a + 2c)\,X^2 + (b + 2d)X^3$$

Associated matrix : $M^L_{4,3} = \begin{bmatrix} a + 0c & 0 & 0 \\ b + 0d & a + 1c & 0 \\ 0 & b + 1d & a + 2c \\ 0 & 0 & b + 2d \end{bmatrix}$

Scheme of the Algorithm

$$\mathbb{K}[X]_{\leq 4n} \xleftarrow{\;B\;} \mathbb{K}[X]_{\leq 3n} \xleftarrow{\;A\;} \mathbb{K}[X]_{\leq 2n}$$

$$C = BA$$

$$M^C = M^B_{4n,3n} \times M^A_{3n,2n}$$

Algorithm in Action when $n = 1$

Input :

$B = (e + f\,X) + (g + h\,X)\theta,\ A = (a + b\,X) + (c + d\,X)\theta$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

gives the multipoint-evaluation of polynomials.

Output :

$C = C_0(\theta) + XC_1(\theta) + X^2C_2(\theta)$

Algorithm in Action when $n = 1$

Input :

$B = (e + f\,X) + (g + h\,X)\theta,\ A = (a + b\,X) + (c + d\,X)\theta$

We get
$$M_{4,3}^{A} = \begin{bmatrix} a & 0 & 0 \\ b & a+c & 0 \\ 0 & b+d & a+2c \\ 0 & 0 & b+2d \end{bmatrix}$$

Output :

$C = C_0(\theta) + XC_1(\theta) + X^2C_2(\theta)$

Algorithm in Action when $n = 1$

Input :

$B = (e + f\,X) + (g + h\,X)\theta,\ A = (a + b\,X) + (c + d\,X)\theta$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

gives the multipoint-evaluation of polynomials.

Output :

$C = C_0(\theta) + XC_1(\theta) + X^2C_2(\theta)$

Algorithm in Action when $n = 1$

Input :

$B = (e + f\,X) + (g + h\,X)\theta,\ A = (a + b\,X) + (c + d\,X)\theta$

We get
$$M^B_{5,4} = \begin{bmatrix} e & 0 & 0 & 0 \\ f & e+g & 0 & 0 \\ 0 & f+h & e+2g & 0 \\ 0 & 0 & f+2h & e+3g \\ 0 & 0 & 0 & f+3h \end{bmatrix}$$

Output :

$C = C_0(\theta) + XC_1(\theta) + X^2C_2(\theta)$

Algorithm in Action when $n = 1$

Input :

$B = (e + f\,X) + (g + h\,X)\theta,\ A = (a + b\,X) + (c + d\,X)\theta$

$$M^C = M^B_{5,4}\,M^A_{4,3}$$

Output :

$C = C_0(\theta) + XC_1(\theta) + X^2C_2(\theta)$

Algorithm in Action when $n = 1$

Input :

$B = (e + f\,X) + (g + h\,X)\theta,\ A = (a + b\,X) + (c + d\,X)\theta$

$$M^C = \begin{bmatrix} C_0(0) & 0 & 0 \\ C_1(0) & C_0(1) & 0 \\ C_2(0) & C_1(1) & C_0(2) \\ 0 & C_2(1) & C_1(2) \\ 0 & 0 & C_2(2) \end{bmatrix}$$

Output :

$C = C_0(\theta) + XC_1(\theta) + X^2C_2(\theta)$

Algorithm in Action when $n = 1$

Input :

$B = (e + f\,X) + (g + h\,X)\theta,\ A = (a + b\,X) + (c + d\,X)\theta$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} C_0(0) & C_1(0) & C_2(0) \\ C_0(1) & C_1(1) & C_2(1) \\ C_0(2) & C_1(2) & C_2(2) \end{bmatrix}$$

gives the coefficients of C

Output :

$C = C_0(\theta) + XC_1(\theta) + X^2C_2(\theta)$

Complexity estimates

- Vandermonde and inverse: $\mathcal{O}(n^2)$;
- Matrix product: n^ω, $2 \leq \omega \leq 3$.
    - naive product: $\omega = 3$;
    - Strassen algorithm: $\omega < 2.8$;
    - Coppersmith & Winograd algorithm: $\omega < 2.376$.
- van der Hoeven's algorithm: $37n^\omega + \mathcal{O}(n^2)$.
    - Evaluation: $(2+3)n^\omega + \mathcal{O}(n^2)$;
    - M^C: $24n^\omega$;
    - Interpolation: $8n^\omega + \mathcal{O}(n^2)$.

Extension to $\mathbb{K}[X, X^{-1}]\langle\theta\rangle$

$B = \sum_{i=0}^{n} \sum_{j=-n}^{n} b_{i,j} X^j \theta^i$: idem for A.

$$\begin{tikzcd}
\begin{array}{c}\mathbb{K}[X]_{\leq 4n} \\ \cup \\ \mathbb{K}[X^{-1}]_{\leq 2n}\end{array} & \begin{array}{c}\mathbb{K}[X]_{\leq 3n} \\ \cup \\ \mathbb{K}[X^{-1}]_{\leq n}\end{array} \arrow[l, "B"'] & \mathbb{K}[X]_{\leq 2n} \arrow[l, "A"'] \arrow[ll, bend right, "C = BA"']
\end{tikzcd}$$

$$M^C = 6n \left\{ \overbrace{\qquad\qquad}^{4n} \right. \quad \times \quad 4n \left\{ \overbrace{\qquad\qquad}^{2n} \right.$$

Conversion between $\mathbb{K}[X]\langle\partial\rangle$ and $\mathbb{K}[X,X^{-1}]\langle\theta\rangle$

$$L \in \mathbb{K}[X]\langle\partial\rangle\colon\ L = \sum_{i=0}^{n}\sum_{j=0}^{n} l_{i,j}X^j\partial^i \longleftrightarrow L = \sum_{i=0}^{n}\sum_{j=-n}^{n} \tilde{l}_{i,j}X^j\theta^i.$$

- Write $L = \sum_{i=0}^{n}\sum_{j=0}^{n} l_{i,j}X^{j-i}(X^i\partial^i)$;
- Apply the change of bases:

$$\begin{bmatrix} 1 \\ X\partial \\ \vdots \\ X^n\partial^n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & \dots & 0 \\ \star & 1 & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ \star & \dots & \star & 1 \end{bmatrix}}_{\textit{Construction in } \mathcal{O}(n^2) \textit{ ops}} \begin{bmatrix} 1 \\ \theta \\ \vdots \\ \theta^n \end{bmatrix}$$

- Return $L = \sum_{i=0}^{n}\sum_{j=-n}^{n} \tilde{l}_{i,j}X^j\theta^i$.

Statement of the Result

Theorem (Bostan, Chyzak & Le Roux 2008)

Computing the product $C = BA$ where B, A in $\mathbb{K}[X]\langle\theta\rangle$ (resp. $\mathbb{K}[X]\langle\partial\rangle$) with bidegree (n, n) is equivalent to computing the product MN where M, N are $n \times n$ matrices with entries in $\mathbb{K}$.

Already proved. $C = BA$ reduces to a constant number of products NM plus $\mathcal{O}(n^2)$ other operations in $\mathbb{K}$ [van der Hoeven 2002].
To prove the converse. Interpolation, Evaluation, Conversion tasks can be done efficiently : $\tilde{\mathcal{O}}(n^2)$.

Improving Evaluation, Interpolation and conversion Steps

Evaluation and interpolation of $P = P_0 + \dots P_d X^d \in \mathbb{K}[X]$ at $(d+1)$ distincts points of $\mathbb{K}$: $\mathcal{O}(M(d)\log(d))$ operations in $\mathbb{K}$. It reduces to $\tilde{\mathcal{O}}(d)$ using fast multiplication.

- Evaluation matrices of B and A: $\mathcal{O}(nM(n)\log(n))$ ops in $\mathbb{K}$.
- Computation of C from M^C: $\mathcal{O}(nM(n)\log(n))$ ops in $\mathbb{K}$.

Theorem (Bostan & Schost 2005)

Conversions $L = \sum_{i=0}^{n}\sum_{j=0}^{n} l_{i,j} X^j \partial^i \longleftrightarrow L = \sum_{i=0}^{n}\sum_{j=-n}^{n} \tilde{l}_{i,j} X^j \theta^i$ *can be done using* $2nM(n)\log(n) + \mathcal{O}(nM(n))$ *operations in* $\mathbb{K}$.

Sketch of the proof

- Product of $(n+1) \times (n+1)$ matrices: constant number of products of lower triangular matrices;
- Assume M, N are $(n+1) \times (n+1)$ lower triangular matrices. We construct from M (resp. N) B (resp. A) in $\mathbb{K}[X]\langle\theta\rangle$ with bidegree (n, n): $\tilde{O}(n^2)$ ops in $\mathbb{K}$;

$$\tilde{M} = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow{\text{Interpolation}} B$$

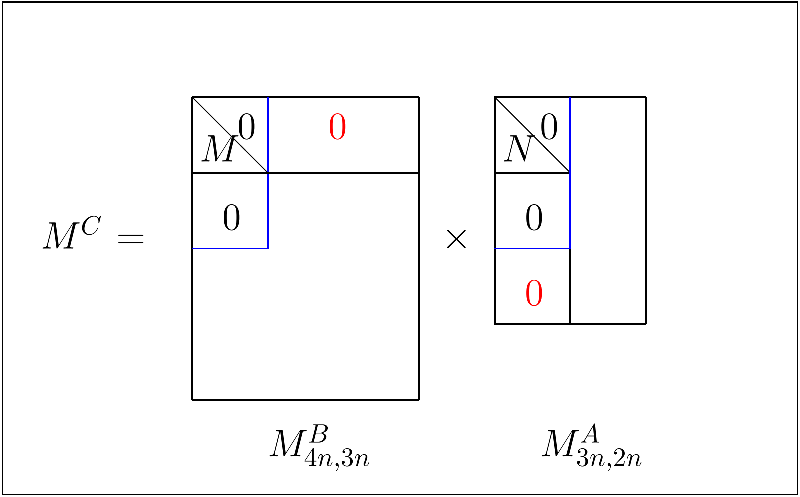
Sketch of the proof

- Product of $(n+1) \times (n+1)$ matrices: constant number of products of lower triangular matrices;
- Assume M, N are $(n+1) \times (n+1)$ lower triangular matrices. We construct from M (resp. N) B (resp. A) in $\mathbb{K}[X]\langle\theta\rangle$ with bidegree (n, n): $\tilde{\mathcal{O}}(n^2)$ ops in $\mathbb{K}$;
- Compute the product $C = BA$ (by one of existing algorithms);
- Compute M^C: $\tilde{\mathcal{O}}(n^2)$ ops in $\mathbb{K}$.

Claim. MN is the upper left $(n+1) \times (n+1)$ submatrix of M^C.
Consequently: MN is computed by one product $C = BA$ in $\mathbb{K}[X]\langle\theta\rangle$ plus $\tilde{\mathcal{O}}(n^2)$ other operations in $\mathbb{K}$.

Sketch of the proof

Proof of the claim



An Interpolation Result

Let $L \in \mathbb{K}[X]\langle\partial\rangle$ with bidegree (n, n).
Define a $\mathbb{K}$-linear endomorphism of $\mathbb{K}[X]_{\leq n}$ by $P \mapsto L(P) \bmod X^{n+1}$.
Denote by M^L the corresponding matrix.

Theorem (Bostan, Chyzak & Le Roux 2008)

$L = \sum_{i=0}^{n} \sum_{j=0}^{n} l_{i,j} X^j \partial^i$ *is uniquely determined by* M^L.

Sketch of the Proof

$$\text{Define} \quad C^L = \begin{bmatrix} l_{0,0} & \cdots & \cdots & \cdots & l_{n,0} \\ \vdots & \ddots & & & \vdots \\ {\color{red} l_{0,\ell}} & & \ddots & & \vdots \\ \vdots & \ddots & & \ddots & \\ l_{n,0} & \cdots & {\color{red} l_{n-\ell,n}} & \cdots & l_{n,n} \end{bmatrix}.$$

We associate to each diagonal a polynomial the coefficients of which in the falling factorials are the entries of the diagonal: for instance when $0 \le \ell \le n$,

$$\mathcal{L}_\ell(T) = \sum_{j=0}^{n-\ell} l_{j,j+\ell} T(T-1)\dots(T-j+1).$$

Sketch of the Proof

Continued ...

Why these polynomials $\mathcal{L}_\ell$?
They appear from the following observation ($0 \le \ell \le n$):

$$X^i \partial^i (X^k) = \frac{k!}{(k-i)!} X^k = \theta\,(\theta - 1)\,\ldots\,(\theta - i + 1)(X^k).$$

Thus $X^i \partial^i = \theta\,(\theta - 1)\,\ldots\,(\theta - i + 1)$.
It follows that L can be rewritten as:

$$L = \sum_{\ell=0}^{n} X^\ell\,\mathcal{L}_\ell(\theta) + \sum_{\ell=-n}^{-1} \mathcal{L}_\ell(\theta)\,\partial^\ell.$$

Sketch of the Proof

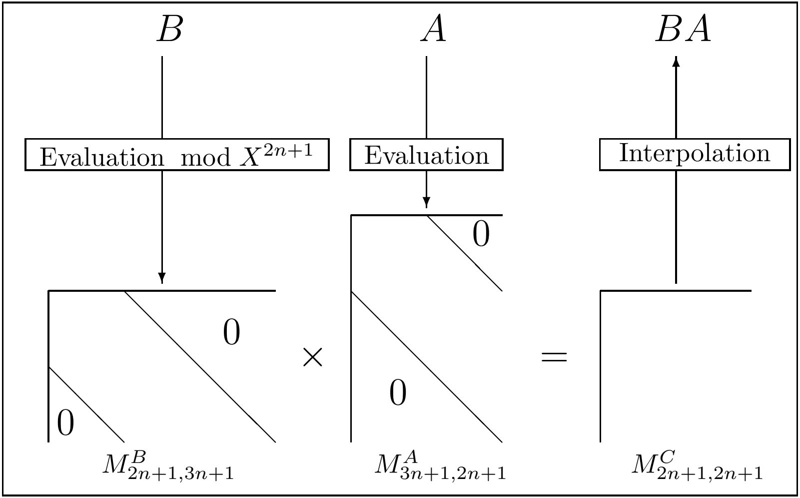
Ended. Whew!

$$C^L = \begin{bmatrix} l_{0,\ell} & & \\ & \ddots & \\ \cdots & & l_{n-\ell,n} \end{bmatrix}$$

$$\updownarrow$$

$$M^L = \begin{bmatrix} \mathcal{L}_\ell(0) & & \\ & \ddots & \\ & & \mathcal{L}_\ell(n-\ell+1) \end{bmatrix}$$

Scheme of the Algorithm



Fast Computation and Interpolation of the Evaluation Matrix

$L \in \mathbb{K}[X]\langle\partial\rangle$ with bidegree (n, n).
Computation of M^L. Its entries are computed diagonal by diagonal.
Assume for instance $0 \leq \ell \leq n$.
We define $\tilde{\mathcal{L}}_\ell(T) = \sum_{i=0}^{n-\ell} l_{i,i+\ell} T^i$.
Claim. the entries of the ℓth subdiagonal of M^L are the coefficients of the truncated series at order $n - \ell + 1$ of:

$$(\tilde{\mathcal{L}}_\ell(T) \exp(T)) \odot (\sum_{k \geq 0} k! T^k).$$

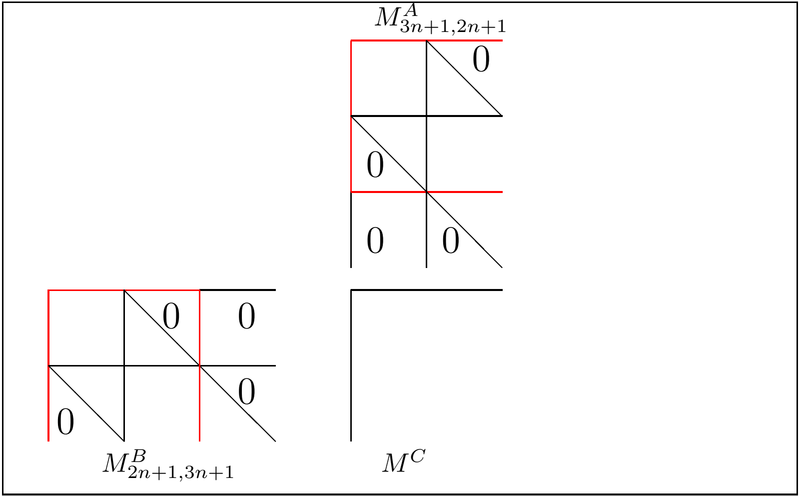
Complexity Estimates

Theorem (Bostan, Chyzak & Le Roux 2008)

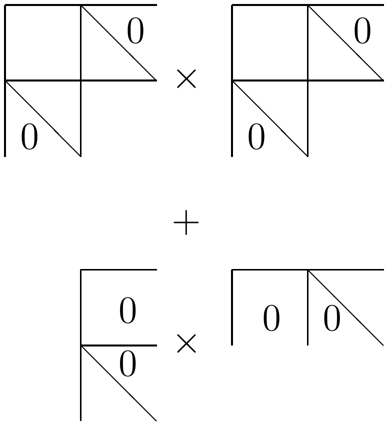
The product BA where $B, A \in \mathbb{K}[X]\langle\partial\rangle$ with bidegree (n, n) can be done in $8\,n^{\omega} + \mathcal{O}(nM(n))$ operations in $\mathbb{K}$.

- Evaluation matrices of B and A : $\mathcal{O}(nM(n))$;
- Computation of M^C : $8\,n^{\omega}$;
- Recover C from M^C : $\mathcal{O}(nM(n))$.

Computation of M^C in $8\,n^\omega$ Operations in $\mathbb{K}$



Computation of M^C in $8\,n^\omega$ Operations in $\mathbb{K}$



cost: $7\,n^\omega$

cost: n^ω

Experimental Issues

n	S2	vdH	Iter	Tak	Sq	Rect	Interp
80	0.50	1.08	2.25	5.61	0.01	0.08	1.10
160	2.24	4.52	19.07	67.73	0.07	0.35	9.22
320	12.2	24.4	187	926	0.27	1.63	75.6
640	88.1	172	2604	∞	1.26	9.40	770
1280	1961	∞	∞	∞	7.24	59.1	∞
80	9.93	28.4	6.99	24.3	0.01	0.07	0.93
160	128	498	118	725	0.05	0.27	6.89
320	2164	∞	2492	∞	0.24	4.37	51.4

Computation modulo 4294967291 and over $\mathbb{Q}$.

Small Positive Characteristic p

Important. $\theta X^p = X^p \theta$.

Theorem (Bostan, Chyzak & Le Roux 2008)

Let $B, A \in \mathbb{K}[X]\langle\theta\rangle$ with bidegree (n, n). Then BA can be computed using $\tilde{\mathcal{O}}(pn^2)$ operations in $\mathbb{K}$.

- Write A and B under the form $A = \sum_{v=0}^{p-1} A_v(X^p, \theta)X^v$ and $B = \sum_{u=0}^{p-1} X^u B_u(X^p, \theta)$. $\mathcal{O}(nM(n)\log(n))$ ops in $\mathbb{K}$;
- Compute the commutative bivariate products with bidegree $(n/p, n)$ $\quad C_{u,v} = A_v B_u$, for $0 \leq u, v < p$.
  $p^2 M(n^2/p)$ ops in $\mathbb{K}$;
- Return $\sum_{u,v=0}^{p-1} X^u C_{u,v}(X^p, \theta)X^v$ in canonical form.

Unbalanced Bidegree Case

$B, A \in \mathbb{K}[X]\langle\partial\rangle$ with bidegree (d, r).

case when $d << r$. For instance $r = d^2$.
Improved vdH and ours: constant number of $d^2 \times d^2$ by $d^2 \times n$ matrix products. $\mathcal{O}(d^{\omega+2})$ ops in $\mathbb{K}$.
Iterative scheme (slightly modified): $\tilde{\mathcal{O}}(d^4)$ ops in $\mathbb{K}$.

case when $d >> r$. For instance $d = r^2$.

- Improved vdH: constant number of $r^2 \times r^2$ by $r^2 \times r^2$ matrix products. $\mathcal{O}(r^{2\omega})$ ops in $\mathbb{K}$.
- Ours: constant number of $r \times r^2$ by $r^2 \times r^2$ matrix products. $\mathcal{O}(r^{\omega+2})$ ops in $\mathbb{K}$.

Iterative scheme: $\tilde{\mathcal{O}}(r^4)$ ops in $\mathbb{K}$.

Product of Sparse Linear Differential Operators

The discussion derives from the following case.

Special case

$\partial^i A$ where $0 \leq i \leq n$, A dense with bidegree (n, n)

- Compute $M^A_{n+1,2n+1}$;
  $\mathcal{O}(nM(n))$ operations in $\mathbb{K}$.
- Compute the ith derivative of each column of $M^A_{n+1,2n+1}$;
  $\mathcal{O}(n^2)$ operations in $\mathbb{K}$. It gives $M^{\partial^i A}$.
- Recover $\partial^i A$ from $M^{\partial^i A}$;
  $\mathcal{O}(nM(n))$